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CERTAIN GEOMETRIC PROPERTIES OF ℓ-HYPERGEOMETRIC FUNCTION

K.V. Vidyasagar

Department of Mathematics, Dr. B.R. Ambedkar Open University, Hyderabad, 500033, Telangana, India

Abstract. Geometric function theory is the branch of complex analysis which deals with the geometric properties of analytic functions. It was founded around the turn of the 20th century and has remained one of the active fields of the current research. In this paper, we study certain geometric properties like κ-uniformly convexity and κ-starlikeness of ℓ-Hypergeometric function and then we prove Alexander transform of ℓ-Hypergeometric function is starlike.

Keywords: Univalent function, close-to-convex function, κ-uniformly iconvexity, κ-starlikeness, Alexander transform, ℓ-Hypergeometric ifunction.

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Corresponding author: Vidyasagar, Kuparala Venkata, Department of Mathematics, Dr. B.R. Ambedkar Open University, Hyderabad, Telangana, India, Tel.: 91 8247685902, e-mail: *vidyavijaya08@gmail.com*

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1 Introduction

Let $\mathcal{A}(\mathbb{D}_1(0))$ denote the class of analytic functions in the open unit disk $\mathbb{D}_1(0) = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{C}$ be the class of all functions $f \in \mathcal{A}(\mathbb{D}_1(0))$ which are normalized by $f(0) = 0$ and $f'(0) = 1$ and have the form (Maharana et al., 2018; Mehrez, 2019; Oluwayemi & Faisal, 2021; Ponnusamy et al., 2011; Ponnusamy & Vuorinen, 2001; Prajapat, 2014, 2011; Purohit, 2012; Vidyasagar, 2020)

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}_1(0). \tag{1}$$

Two functions $f, g \in \mathcal{A}(\mathbb{D}_1(0))$ we say that f is subordinated to g in $\mathbb{D}_1(0)$ and express symbolically $f(z) \prec g(z)$, if there exists a function $\omega \in \mathcal{A}(\mathbb{D}_1(0))$ with $|\omega(z)| < |z|$, $z \in \mathbb{D}_1(0)$ such that $f(z) = g(\omega(z))$ in $\mathbb{D}_1(0)$. Furthermore, if function f is univalent in $\mathbb{D}_1(0)$, then g is subordinate to f provided $g(0) = f(0)$ and $g(\mathbb{D}_1(0)) \subset f(\mathbb{D}_1(0))$. By $\mathcal{S}$ we denote the class of all functions in $\mathcal{C}$ which are univalent in $\mathbb{D}_1(0)$. Let $\mathcal{S}^*(\varepsilon)$, $\mathcal{C}(\varepsilon)$, $\mathcal{K}(\varepsilon)$, $\widetilde{\mathcal{S}}^*(\varepsilon)$ and $\widetilde{\mathcal{C}}(\varepsilon)$ denote the classes of starlike, convex, close-to-convex, strongly starlike and strongly convex functions of order ε, respectively, and are defined as

$$\mathcal{S}^*(\varepsilon) = \left\{ f \in \mathcal{C} : \mathcal{R}e\left(\frac{zf'(z)}{f(z)}\right) > \varepsilon,\ z \in \mathbb{D}_1(0),\ 0 \le \varepsilon < 1 \right\},$$

$$\mathcal{C}(\varepsilon) = \left\{ f \in \mathcal{C} : \mathcal{R}e\left(\frac{(zf'(z))'}{f'(z)}\right) > \varepsilon,\ z \in \mathbb{D}_1(0),\ 0 \le \varepsilon < 1 \right\},$$

$$\mathcal{K}(\varepsilon) = \left\{ f \in \mathcal{C} : \mathcal{R}e\left(\frac{zf'(z)}{g(z)}\right) > \varepsilon,\ z \in \mathbb{D}_1(0),\ 0 \le \varepsilon < 1,\ g \in \mathcal{S}^*(0) \equiv \mathcal{S}^* \right\},$$

$$\widetilde{\mathcal{S}}^*(\varepsilon) = \left\{ f \in \mathcal{C} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\varepsilon\pi}{2},\ z \in \mathbb{D}_1(0),\ 0 \le \varepsilon < 1 \right\},$$

and

$$\widetilde{\mathcal{C}}(\varepsilon) = \left\{ f \in \mathcal{C} : \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\varepsilon\pi}{2}, \quad z \in \mathbb{D}_1(0), \quad 0 \le \varepsilon < 1 \right\}.$$

For more details regarding these classes see Duren (1983); Goodman (1983).

For $z \in \mathbb{C}$, the ℓ-Hypergeometric function is defined as

$$H\begin{bmatrix} a; & & z \\ & b;\ (c:\ell); & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n (c)_n^{\ell n}} \frac{z^n}{n!}, \tag{2}$$

where $(\gamma)_n = \Gamma(\gamma+n)/\Gamma(\gamma)$, $a,\ \ell \in \mathbb{C}$ with $\mathcal{R}e(\ell) \ge 0$ and $b,\ c \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$. If we put $\ell = 0$ in (2), then ℓ-H function turns to well known confluent hypergeometric function,

$$H\begin{bmatrix} a; & & z \\ & b;\ (c:0); & \end{bmatrix} = {}_1F_1\begin{bmatrix} a; & z \\ b; & \end{bmatrix}. \tag{3}$$

The ℓ-H function (2) was recently studied in Chudasama & Dev (2016).

We note that ℓ-Hypergeometric function (2) does not belong to the family $\mathcal{C}$. Thus, it is natural to consider the following normalization of ℓ-H function:

$$\begin{aligned} \mathcal{H}(a;b;(c,\ell);z) &= zH\begin{bmatrix} a; & & z \\ & b;\ (c:\ell); & \end{bmatrix} \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}} \frac{z^n}{(n-1)!}. \end{aligned} \tag{4}$$

Motivated by above works, in this paper we study certain geometric properties like κ-uniformly convexity and κ-starlikeness of ℓ-Hypergeometric function and then we prove Alexander transform of ℓ-Hypergeometric function is starlike. Let $\kappa - \mathcal{UCV}$ and $\kappa - \mathcal{ST}$ be the subclasses of $\mathcal{S}$ consisting of functions which are κ-uniformly convex and κ-starlike, respectively (Kanas & Wisniowska, 1999, 2000). They are given by,

$$\begin{aligned} \kappa - \mathcal{UCV} = \Bigg\{ f \in \mathcal{S} : \mathcal{R}e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \kappa\left|\frac{zf''(z)}{f'(z)}\right|, \\ z \in \mathbb{D}_1(0), \kappa \ge 0 \Bigg\}, \end{aligned} \tag{5}$$

$$\begin{aligned} \kappa - \mathcal{ST} = \Bigg\{ f \in \mathcal{S} : \mathcal{R}e\left(\frac{zf'(z)}{f(z)}\right) > \kappa\left|\frac{zf'(z)}{f(z)} - 1\right|, \\ z \in \mathbb{D}_1(0), \kappa \ge 0 \Bigg\}. \end{aligned} \tag{6}$$

The class of all functions $p \in \mathcal{A}(\mathbb{D}_1(0))$ with $p(0) = 1$ satisfying the condition

$$\mathcal{R}e\, p(z) > \varepsilon,\ z \in \mathbb{D}_1(0),\ \varepsilon \in [0,1)$$

be denoted by $\mathcal{P}(\varepsilon)$. In particular, $\mathcal{P}(0) = \mathcal{P}$ is the well-known Caratheódory class of functions with positive real part in $\mathbb{D}_1(0)$ (Goodman, 1983). The following lemmas are useful in the next section.

Lemma 1. *(Owa et al., 2002) If $f \in \mathcal{C}$ satisfies the inequality*

$$|zf''(z)| < \frac{1-\varepsilon}{4}, \ z \in \mathbb{D}_1(0), \ \varepsilon \in [0,1), \tag{7}$$

then,

$$\mathcal{R}ef'(z) > \frac{1+\varepsilon}{2}, \ z \in \mathbb{D}_1(0), \ \varepsilon \in [0,1).$$

Lemma 2. *(Silverman, 1975) Let $f \in \mathcal{C}$ and $\varepsilon \in [0,1)$, then*

(i) $f \in \mathcal{S}^*(\varepsilon)$ provided

$$\sum_{n=2}^{\infty}(n-\varepsilon)|a_n| \leq 1-\varepsilon \tag{8}$$

(i) $f \in \mathcal{C}(\varepsilon)$ provided

$$\sum_{n=2}^{\infty}n(n-\varepsilon)|a_n| \leq 1-\varepsilon. \tag{9}$$

Lemma 3. *(Kanas & Wisniowska, 1999, 2000) Let $f \in \mathcal{C}$. If for some $\kappa \geq 0$,*

$$\sum_{n=2}^{\infty}n(n-1)|a_n| \leq \frac{1}{\kappa+2} \tag{10}$$

and

$$\sum_{n=2}^{\infty}[n+\kappa(n-1)]|a_n| \leq 1, \tag{11}$$

then $f \in \kappa-\mathcal{UCV}$ and $f \in \kappa-\mathcal{ST}$, respectively.

2 Main Results

In the sequence, convexity of order ε, close-to-convexity of order $(1+\varepsilon)/2$ for normalized ℓ-Hypergeometric function $\mathcal{H}(a;b;(c,\ell);z)$ are investigated. Certain sufficient conditions for $\mathcal{H}(a;b;(c,\ell);z)$ to be in the classes $\mathcal{P}(\varepsilon), \mathcal{S}^*(\varepsilon)$, $\mathcal{C}(\varepsilon), \kappa-\mathcal{UCV}$ and $\kappa-\mathcal{ST}$ are also given.

Theorem 1. *If $a,b,c,\ell \in \mathbb{R}$ with $0 < a < b$, $\ell \geq 1$ and $c \geq 1+\sqrt{3}$. Then $\mathcal{H}(a;b;(c,\ell);z)$ is starlike in $\mathbb{D}_1(0)$ i.e $\mathcal{H}(a;b;(c,\ell);\cdot) \in \mathcal{S}^*$.*

Proof. Let $p(z)$ be the function defined by

$$p(z) = \frac{z\mathcal{H}'(a;b;(c,\ell);z)}{\mathcal{H}(a;b;(c,\ell);z)}, \quad z \in \mathbb{D}_1(0).$$

Since $\dfrac{\mathcal{H}(a;b;(c,\ell);z)}{z} \neq 0$, the function p is analytic in $\mathbb{D}_1(0)$ and $p(0)=1$. To prove the result, we need to show that $\mathcal{R}e(p(z)) > 0$. Since $c > 1$ and $\ell \geq 1$, it follows that $(c)_n \leq (c)_n^{\ell n}$ for all $n \in \mathbb{N}$. So, from the hypothesis,

$$\begin{aligned}\left|\mathcal{H}'(a;b;(c,\ell);z) - \frac{\mathcal{H}(a;b;(c,\ell);z)}{z}\right| &= \left|\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{z^n}{(n-1)!}\right| \\ &< \sum_{n=1}^{\infty}\frac{1}{(c)_n} \\ &= \sum_{n=1}^{\infty}\frac{1}{c(c+1)(c+2)\cdots(c+n-1)} \\ &< \frac{1}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n} = \frac{c+1}{c^2},\end{aligned} \tag{12}$$

and

$$\begin{aligned}\left|\frac{\mathcal{H}(a;b;(c,\ell);z)}{z}\right| &\geq 1-\left|\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{z^n}{n!}\right| \\ &\geq 1-\sum_{n=1}^{\infty}\frac{1}{(c)_n} \\ &= 1-\sum_{n=1}^{\infty}\frac{1}{c(c+1)(c+2)\cdots(c+n-1)} \\ &> 1-\frac{1}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n}=\frac{c^2-c-1}{c^2}.\end{aligned} \tag{13}$$

From (12) and (13), we have

$$\begin{aligned}\left|\frac{z\mathcal{H}'(a;b;(c,\ell);z)}{\mathcal{H}(a;b;(c,\ell);z)}-1\right| &= \left|\frac{\mathcal{H}'(a;b;(c,\ell);z)-\frac{\mathcal{H}(a;b;(c,\ell);z)}{z}}{\frac{\mathcal{H}(a;b;(c,\ell);z)}{z}}\right| \\ &< \frac{c+1}{c^2-c-1}, \quad z\in\mathbb{D}_1(0).\end{aligned}$$

Since $c\geq 1+\sqrt{3}$, it follows that $\frac{c+1}{c^2-c-1}\leq 1$ and hence $\mathcal{H}(a;b;(c,\ell);z)$ is starlike in $\mathbb{D}_1(0)$. □

Theorem 2. *If $a,b,c,\ell\in\mathbb{R}$ with $0<a<b$, $\ell\geq 1$. For $0\leq\varepsilon<1$, let*

$$\varrho(\varepsilon)=\frac{(2-\varepsilon)+\sqrt{5\varepsilon^2-16\varepsilon+12}}{2(1-\varepsilon)}.$$

If $c\geq\varrho(\varepsilon)$, then $\mathcal{H}(a;b;(c,\ell);z)$ is starlike function of order ε i.e $\mathcal{H}(a;b;(c,\ell);\cdot)\in\mathcal{S}^*(\varepsilon)$.

Proof. Following the proof of Theorem 1, $\mathcal{H}(a;b;(c,\ell);z)$ is starlike function of order ε, if $\dfrac{c+1}{c^2-c-1}\leq 1-\varepsilon$. This is true from the hypothesis. This completes the proof. □

Theorem 3. *If $a,b,c,\ell\in\mathbb{R}$ with $0<a<b$, $\ell\geq 1$. For $0\leq\varepsilon<1$, let*

$$\vartheta(\varepsilon)=\frac{(8-3\varepsilon)+\sqrt{17\varepsilon^2-68\varepsilon+76}}{2(1-\varepsilon)}.$$

If $c\geq\vartheta(\varepsilon)$. Then $\mathcal{H}(a;b;(c,\ell);z)$ is convex of order ε i.e $\mathcal{H}(a;b;(c,\ell);\cdot)\in\mathcal{C}(\varepsilon)$.

Proof. Under the hypothesis, we obtain

$$\begin{aligned}\left|\mathcal{H}'(a;b;(c,\ell);z)\right| &\leq \left|1+\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{(n+1)z^n}{n!}\right| \\ &\leq 1+\sum_{n=1}^{\infty}\frac{n+1}{(c)_n} \\ &= 1+\sum_{n=1}^{\infty}\frac{n}{(c)_n}+\sum_{n=1}^{\infty}\frac{1}{(c)_n} \\ &\leq 1+\frac{1}{c}+\frac{2}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n} \\ &= \frac{c^2+3c+2}{c^2}.\end{aligned} \tag{14}$$

For the reverse inequality, we have

$$\begin{aligned}\left|\mathcal{H}'(a;b;(c,\ell);z)\right| &\geq 1-\left|\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{(n+1)z^n}{n!}\right| \\ &\geq 1-\sum_{n=1}^{\infty}\frac{n+1}{(c)_n} \\ &= 1-\sum_{n=1}^{\infty}\frac{n}{(c)_n}-\sum_{n=1}^{\infty}\frac{1}{(c)_n} \\ &\geq 1-\frac{1}{c}-\frac{2}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n} \\ &= \frac{c^2-3c-2}{c^2}.\end{aligned} \tag{15}$$

From (14) and (15), we obtained

$$\frac{c^2-3c-2}{c^2}\leq\left|\mathcal{H}'(a;b;(c,\ell);z)\right|\leq\frac{c^2+3c+2}{c^2},\ z\in\mathbb{D}_1(0). \tag{16}$$

From (4), we have

$$\begin{aligned}|z\mathcal{H}''(a;b;(c,\ell);z)| &= \left|\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{n(n+1)z^n}{n!}\right| \\ &\leq\sum_{n=1}^{\infty}\frac{n(n+1)}{(c)_n} \\ &\leq\frac{4}{c}+\frac{1}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n} \\ &= \frac{5c+1}{c^2}.\end{aligned} \tag{17}$$

Now, from (16) and (17), we get

$$\left|\frac{z\mathcal{H}''(a;b;(c,\ell);z)}{\mathcal{H}'(a;b;(c,\ell);z)}\right|\leq\frac{5c+1}{c^2-3c-2},\quad z\in\mathbb{D}_1(0).$$

Since $c>\vartheta(\varepsilon)$, it follows that $\dfrac{5c+1}{c^2-3c-2}\leq 1-\varepsilon$. Hence, $\mathcal{H}(a;b;(c,\ell);z)$ is convex of order ε in $\mathbb{D}_1(0)$. □

If we take $\varepsilon=0$ in Theorem 3, then we have the following result.

Corollary 1. *If $a,b,c,\ell\in\mathbb{R}$ with $0<a<b$, $\ell\geq 1$ and $c\geq 4+\sqrt{19}$. Then $\mathcal{H}(a;b;(c,\ell);\cdot)\in\mathcal{C}$.*

Theorem 4. *If $a,b,c,\ell\in\mathbb{R}$ with $0<a<b$, $\ell\geq 1$. For $0\leq\varepsilon<1$, let*

$$\rho(\varepsilon)=\frac{10+2\sqrt{26-\varepsilon}}{1-\varepsilon}.$$

If $c\geq\rho(\varepsilon)$. Then $\mathcal{H}(a;b;(c,\ell);z)$ is close-to-convex of order $\dfrac{1+\varepsilon}{2}$ i.e $\mathcal{H}(a;b;(c,\ell);\cdot)\in\mathcal{K}\left(\dfrac{1+\varepsilon}{2}\right)$.

Proof. Using (17) and Lemma 1, we have

$$|z\mathcal{H}''(a;b;(c,\ell);z)|\leq\frac{5c+1}{c^2},\quad z\in\mathbb{D}_1(0).$$

Since $c \geq \rho(\varepsilon)$, it follows that $\dfrac{5c+1}{c^2} \leq \dfrac{1-\varepsilon}{4}$. this proves that $\mathcal{R}e(\mathcal{H}'(a;b;(c,\ell);z)) > \dfrac{1+\varepsilon}{2}$ and hence $\mathcal{H}(a;b;(c,\ell);\cdot) \in \mathcal{K}\left(\dfrac{1+\varepsilon}{2}\right)$. ☐

If we take $\varepsilon = 0$ in Theorem 4, then we have the following result.

Corollary 2. *If $a,b,c,\ell \in \mathbb{R}$ with $0<a<b$, $\ell \geq 1$ and $c \geq 10+2\sqrt{26}$. Then $\mathcal{H}(a;b;(c,\ell);z)$ is close-to-convex of order $\frac{1}{2}$ i.e $\mathcal{H}(a;b;(c,\ell);\cdot) \in \mathcal{K}\left(\frac{1}{2}\right)$.*

Theorem 5. *If $a,b,c,\ell \in \mathbb{R}$ with $0<a<b$, $\ell \geq 1$. For $0 \leq \varepsilon < 1$, let*

$$\psi(\varepsilon) = \frac{1+\sqrt{5-4\varepsilon}}{2(1-\varepsilon)}.$$

If $c \geq \psi(\varepsilon)$. Then $\dfrac{\mathcal{H}(a;b;(c,\ell);z)}{z} \in \mathcal{P}(\varepsilon)$.

Proof. Let $p(z)$ be the function defined by

$$p(z) = \frac{\mathcal{H}(a;b;(c,\ell);z)/z - \varepsilon}{(1-\varepsilon)}.$$

The function $p(z)$ is analytic in $\mathbb{D}_1(0)$ and $p(0)=1$. To prove the result, we have to show that $|p(z)-1|<1$. If $z \in \mathbb{D}_1(0)$, then

$$\begin{aligned}
|p(z)-1| &= \left| \frac{1}{1-\varepsilon} \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n (c)_n^{\ell n}} \frac{z^n}{n!} \right| \\
&\leq \frac{1}{1-\varepsilon} \sum_{n=1}^{\infty} \frac{1}{(c)_n} \\
&= \frac{1}{1-\varepsilon} \sum_{n=1}^{\infty} \frac{1}{c(c+1)(c+2)\cdots(c+n-1)} \\
&\leq \frac{1}{1-\varepsilon} \frac{1}{c} \sum_{n=0}^{\infty} \frac{1}{(c+1)^n} = \frac{c+1}{c^2(1-\varepsilon)}.
\end{aligned}$$

Since $c \geq \psi(\varepsilon)$, it follows that $\dfrac{c+1}{c^2(1-\varepsilon)} \leq 1$. Hence, $\dfrac{\mathcal{H}(a;b;(c,\ell);z)}{z} \in \mathcal{P}(\varepsilon)$. ☐

If we take $\varepsilon = 0$ in Theorem 5, then we have the following result.

Corollary 3. *If $a,b,c,\ell \in \mathbb{R}$ with $0<a<b$, $\ell \geq 1$ and $c \geq \dfrac{1+\sqrt{5}}{2}$. Then $\dfrac{\mathcal{H}(a;b;(c,\ell);z)}{z} \in \mathcal{P}$.*

Theorem 6. *If $a,b,c,\ell \in \mathbb{R}$ with $0<a<b$, $c \geq 1$ and $\ell \geq 1$. For $0 \leq \varepsilon < 1$,*

$$\mathcal{H}'(a;b;(c,\ell);1) - \varepsilon \mathcal{H}(a;b;(c,\ell);1) \leq 2(1-\varepsilon).$$

Then $\mathcal{H}(a;b;(c,\ell);z) \in \mathcal{S}^*(\varepsilon)$.

Proof. From (4), we have $\mathcal{H}(a;b;(c,\ell);z) = z + \sum_{n=2}^{\infty} A_{n-1} z^n$, where $A_{n-1} = \dfrac{(a)_{n-1}}{(n-1)!(b)_{n-1}(c)_{n-1}^{\ell(n-1)}}$.
Then from the hypothesis, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} (n-\varepsilon)\,|A_{n-1}| &= \sum_{n=2}^{\infty} nA_{n-1} - \varepsilon \sum_{n=2}^{\infty} A_{n-1} \\
&= (\mathcal{H}'(a;b;(c,\ell);z) - 1) - \varepsilon(\mathcal{H}(a;b;(c,\ell);z) - 1) \\
&= \mathcal{H}'(a;b;(c,\ell);z) - \varepsilon \mathcal{H}(a;b;(c,\ell);z) - 1 + \varepsilon \\
&\leq 1 - \varepsilon.
\end{aligned}$$

Hence form Lemma 2, $\mathcal{H}(a; b; (c, \ell); z)$ is a starlike of order ε. □

Theorem 7. *If $a, b, c, \ell \in \mathbb{R}$ with $0 < a < b$, $c \geq 1$ and $\ell \geq 1$. For $0 \leq \varepsilon < 1$,*

$$\mathcal{H}''(a; b; (c, \ell); 1) + (1-\varepsilon)\mathcal{H}'(a; b; (c, \ell); 1) \leq 2(1-\varepsilon).$$

Then $\mathcal{H}(a; b; (c, \ell); z) \in \mathcal{C}(\varepsilon)$.

Proof. From (4), we have $\mathcal{H}(a; b; (c, \ell); z) = z + \sum_{n=2}^{\infty} A_{n-1} z^n$, where $A_{n-1} = \dfrac{(a)_{n-1}}{(n-1)!(b)_{n-1}(c)_{n-1}^{\ell(n-1)}}$.
Then from the hypothesis, we have

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-\varepsilon)\left|A_{n-1}\right| &= \sum_{n=2}^{\infty} n(n-1)A_{n-1} + (1-\varepsilon)\sum_{n=2}^{\infty} A_{n-1} \\ &= \mathcal{H}''(a; b; (c, \ell); 1) + (1-\varepsilon)(\mathcal{H}'(a; b; (c, \ell); 1) - 1) \\ &= \mathcal{H}''(a; b; (c, \ell); 1) + (1-\varepsilon)\mathcal{H}'(a; b; (c, \ell); 1) - (1-\varepsilon) \\ &\leq 1-\varepsilon.\end{aligned}$$

Hence form Lemma 2, $\mathcal{H}(a; b; (c, \ell); z)$ is a convex of order ε. □

Theorem 8. *If $a, b, c, \ell \in \mathbb{R}$ with $0 < a < b$, $c \geq 1$, $\ell \geq 1$ and $\kappa \geq 0$. Then the sufficient condition for $\mathcal{H}(a; b; (c, \ell); z)$ to be in $\kappa - \mathcal{ST}$ is*

$$\mathcal{H}'(a; b; (c, \ell); 1) - \frac{\kappa}{\kappa+1}\mathcal{H}(a; b; (c, \ell); 1) \leq \frac{2}{\kappa+1}.$$

Proof. From (4), we have $\mathcal{H}(a; b; (c, \ell); z) = z + \sum_{n=2}^{\infty} A_{n-1} z^n$, where $A_{n-1} = \dfrac{(a)_{n-1}}{(n-1)!(b)_{n-1}(c)_{n-1}^{\ell(n-1)}}$.
Then from the hypothesis, we have

$$\begin{aligned}\sum_{n=2}^{\infty}[n+\kappa(n-1)]\left|A_{n-1}\right| &= (1+\kappa)\sum_{n=2}^{\infty} nA_{n-1} - \kappa\sum_{n=2}^{\infty} A_{n-1} \\ &= (1+\kappa)(\mathcal{H}'(a; b; (c, \ell); z) - 1) \\ &\qquad - \kappa(\mathcal{H}(a; b; (c, \ell); z) - 1) \\ &= (1+\kappa)\mathcal{H}'(a; b; (c, \ell); z) \\ &\qquad - \kappa\mathcal{H}(a; b; (c, \ell); z) - 1 \\ &\leq 1.\end{aligned}$$

Hence form Lemma 3, $\mathcal{H}(a; b; (c, \ell); z) \in \kappa - \mathcal{ST}$. □

Theorem 9. *If $a, b, c, \ell \in \mathbb{R}$ with $0 < a < b$, $c \geq 1$, $\ell \geq 1$ and $\kappa \geq 0$. Then the sufficient condition for $\mathcal{H}(a; b; (c, \ell); z)$ to be in $\kappa - \mathcal{UCV}$ is*

$$\mathcal{H}''(a; b; (c, \ell); 1) \leq \frac{1}{\kappa+2}.$$

Proof. From (4), we have $\mathcal{H}(a; b; (c, \ell); z) = z + \sum_{n=2}^{\infty} A_{n-1} z^n$, where $A_{n-1} = \dfrac{(a)_{n-1}}{(n-1)!(b)_{n-1}(c)_{n-1}^{\ell(n-1)}}$.
Then from the hypothesis, we have

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)\left|A_{n-1}\right| &= \sum_{n=2}^{\infty} n(n-1)A_{n-1} \\ &= \mathcal{H}''(a; b; (c, \ell); 1) \\ &\leq \frac{1}{\kappa+2}.\end{aligned}$$

Hence form Lemma 3, $\mathcal{H}(a; b; (c, \ell); z) \in \kappa - \mathcal{UCV}$. □

For a function $f \in \mathcal{C}$ given by (1), the Alexander transform $A(f) : \mathbb{D}_1(0) \to \mathbb{C}$ is defined by (see Alexander (2015))

$$A(f)z = \int_0^z \frac{f(w)}{w} dw = z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n.$$

Theorem 10. *If $a, b, c, \ell \in \mathbb{R}$ with $0 < a < b$, $c \geq 1$ and $\ell \geq 1$. Then the sufficient condition for $A(\mathcal{H}(a; b; (c, \ell); z))$ to be in the class $\mathcal{S}^*$ is $\mathcal{H}(a; b; (c, \ell); 1) \leq 2$.*

Proof. From (4), we have

$$\frac{\mathcal{H}(a; b; (c, \ell); z)}{z} = 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}} \frac{z^{n-1}}{(n-1)!} = 1 + \sum_{n=2}^{\infty} A_{n-1} z^{n-1},$$

where

$$A_{n-1} = \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}(n-1)!}.$$

Thus,

$$\begin{aligned} A(\mathcal{H}(a; b; (c, \ell); z)) &= \int_0^z \frac{\mathcal{H}(a; b; (c, \ell); w)}{w} dw \\ &= z + \sum_{n=2}^{\infty} A_{n-1} \frac{z^n}{n} = \sum_{n=1}^{\infty} a_{n-1} z^n, \end{aligned}$$

where $a_1 = 1,\ a_n = \dfrac{A_{n-1}}{n},\ n \geq 2$. From Lemma 2, we have $A(\mathcal{H}(a; b; (c, \ell); z)) \in \mathcal{S}^*(0) = \mathcal{S}^*$ if,

$$\sum_{n=2}^{\infty} n|a_n| \leq 1.$$

That is

$$\begin{aligned} \sum_{n=2}^{\infty} n|a_n| &= \sum_{n=2}^{\infty} n \frac{A_{n-1}}{n} \\ &= \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}(n-1)!} \\ &= \mathcal{H}(a; b; (c, \ell); 1) - 1 \leq 1. \end{aligned}$$

Which is true, since $\mathcal{H}(a; b; (c, \ell); 1) \leq 2$. This completes the proof. □

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Permanence and Uniformly Asymptotic Stability of Almost Periodic Positive Solutions for a Dynamic Commensalism Model on Time Scales

K. R. Prasad[a], M. Khuddush[a,*] and K. V. Vidyasagar[a,b]

[a] *Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, India-530003,*
[b] *Department of Mathematics, Government Degree College for Women, Marripalem, Koyyuru Mandal, Visakhapatnam, India-531116.*

Abstract. In this paper, we study dynamic commensalism model with nonmonotic functional response, density dependent birth rates on time scales and derive sufficient conditions for the permanence. We also establish the existence and uniform asymptotic stability of unique almost periodic positive solution of the model by using Lyapunov functional method.

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1. Introduction

Ecology relates to the study of living beings in connection to their living styles. Research in the area of theoretical ecology was first studied by Volterra [29] and Lotka [23]. Later many ecologists and mathematicians contributed to the growth of this area of knowledge as reported in [3, 7, 12, 24, 25] and references therein.

*Corresponding author. Email: khuddush89@gmail.com

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The ecological interactions can be broadly classified as prey-predator, competition, commensalism, ammensalism, and neutralism etc.

A two species Commensalisms is an ecological connection between two species where one species X gain benefits while those of the other species Y neither benfit nor harmed. Here, X may referred as the commensal species while Y the host. Some examples are Cattle Egret, Anemonetish and Barnacles etc. The host species Y supports the commensal species X which has a natural growth rate in spite of a support other than from X. The commensal species X, in spite of the limitation of its natural resources flourishes drawing strength from the host species Y. The model is characterized by a system of first order nonlinear differential equations. In the last decades, commensalism model studied many researchers [8, 9, 19, 20, 32].

Chen at el. [6] proposed the following two species commensal symbiosis models with nonmonotonic functional response,

$$\begin{aligned} u_1'(t) &= u_1(t)\left[a_{11} - b_{12}u_1(t) + \frac{cu_2(t)}{d + u_2^2(t)}\right], \\ u_2'(t) &= u_2(t)\left[a_{21} - b_{22}u_2(t)\right], \end{aligned}$$

where $a_{11}, a_{21}, b_{12}, b_{22}, c, d$ are all positive constants and showed that the system admits a unique globally asymptotically stable positive equilibrium.

Zhao et al. [35] proposed and analyzed a commensalism model with nonmonotonic functional response and density-dependent birth rates,

$$\left.\begin{aligned} u_1'(t) &= u_1(t)\left[\frac{a_{11}}{a_{12} + a_{13}u_1(t)} - a_{14} - b_1u_1(t) + \frac{cu_2(t)}{d + u_2^2(t)}\right], \\ u_2'(t) &= u_2(t)\left[\frac{a_{21}}{a_{22} + a_{23}u_2(t)} - a_{24} - b_2u_2(t)\right], \end{aligned}\right\} \tag{1}$$

where a_{ij} (i = 1, 2, j = 1, 2, 3, 4) and $b_1, c, d,$ and b_2 are all positive constants. Here $u_1(t)$ and $u_2(t)$ are the densities of the first and second species at time t, respectively. a_{11} and a_{21} stand for the total resources available per unit time for species u and v, respectively. By applying the differential inequality theory, they showed that each equilibrium can be globally attractive under suitable conditions.

Xie et al. [33] derived sufficient conditions for the existence of positive periodic solution of the following discrete Lotka-Volterra commensal symbiosis model

$$\begin{aligned} u(k+1) &= u(k)\exp\left\{a_1(k) - b_1(k)u(k) + c_1(k)v(k)\right\} \\ v(k+1) &= v(k)\exp\left\{a_2(k) - b_2(k)v(k)\right\} \end{aligned}$$

where $\{b_i(k)\}$, $i = 1, 2$, $\{c_i(k)\}$ are all positive ω-periodic sequences, ω is a fixed positive integer, $\{a_i(k)\}$, are ω-periodic sequences such that $\overline{a}_i = \frac{1}{\omega}\sum_{k=0}^{\omega-1} a_i(k) > 0$, $i = 1, 2$.

The differential, difference and dynamic equations on time scales are three equations play important role for modelling in the environment. Among them, the theory of dynamic equations on time scales is the most recent and was introduced by Stefan Hilger in his PhD thesis in 1988 with three main features: unification, extension and discretization. Since a time scale is any closed and nonempty subset of the real numbers set. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles [1, 2] and monographs of Bohner and Peterson [4, 5]. In the real world phenomena, since the almost periodic variation of the environment plays a crucial role in many biological and ecological dynamical systems and is more frequent and general than the periodic variation of the environment. In this paper we systematically unify the existence of almost periodic solutions of commensalism model with nonmonotic functional response and density dependent birth rates modelled by ordinary differential equations and their discrete analogues in the form of difference equations and to extend these results to more general time scales. The concept of almost periodic time scales was proposed by Li and Wang [13]. Based on this concept, some works have been done (see [14–18, 21, 22, 26, 28] and references therein).

Recently, Wang [30] established a criteria for global existence of multiple periodic solutions to the dynamic predator-prey model with delays,

$$
\begin{aligned}
u_1^{\Delta}(t) &= a(t) - b(t)\exp\{u_1(t)\} - \frac{c(t)\exp\{2u_2(t)\}}{m^2\exp\{2u_2(t)\} + \exp\{2u_1(t)\}} - h(t)\exp\{-u_1(t)\},\\
u_2^{\Delta}(t) &= \frac{f(t)\exp\{u_1(t-\tau(t)) + u_2(t-\tau(t))\}}{m^2\exp\{2u_2(t-\tau(t))\} + \exp\{2u_1(t-\tau(t))\}} - d(t),
\end{aligned}
$$

by applying continuation theorem based on Gaines and Mawhin's coincidence degree theory, and the corresponding discrete system was studied by [11].

Wang et al. [31] considered the following competitive system on time scales,

$$
\begin{aligned}
u_1^{\Delta}(t) &= r_1(t) - a_1(t)\exp\{u_1(t)\} - \frac{b_1(t)\exp\{u_2(t)\}}{1+\exp\{u_2(t)},\\
u_2^{\Delta}(t) &= r_2(t) - a_2(t)\exp\{u_2(t)\} - \frac{b_2(t)\exp\{u_1(t)\}}{1+\exp\{u_1(t)}.
\end{aligned}
$$

and established existence and uniformly asymptotic stability of unique positive almost periodic solutions by time scale calculus theory and Lyapunov functional method

Prasad et al. [27] studied the following 3-species predator-prey competition model on time scales,

$$
\begin{aligned}
u_1^{\Delta}(t) &= r_1(t) - \exp\{u_1(t)\} - \alpha\exp\{u_2(t)\} - \beta\exp\{u_3(t)\},\\
u_2^{\Delta}(t) &= r_2(t) - \beta\exp\{u_1(t)\} - \exp\{u_2(t)\} - \alpha\exp\{u_3(t)\},\\
u_3^{\Delta}(t) &= r_3(t) - \alpha\exp\{u_1(t)\} - \beta\exp\{u_2(t)\} - \exp\{u_3(t)\},
\end{aligned}
$$

and established sufficient conditions for the existence and uniform asymptotic stability of unique positive almost periodic solution of system.

Motivated by the aforementioned reasons in this paper we study commensalism model with nonmonotic functional response and density dependent birth rates on time scales,

$$\left.\begin{aligned}\omega_1^{\Delta}(t) =&\frac{a_{11}(t)}{a_{12}(t)+a_{13}(t)\exp\{\omega_1(t)\}} - a_{14}(t) - b_1(t)\exp\{\omega_1(t)\}\\ &+\frac{c(t)\exp\{\omega_2(t)\}}{d(t)+\exp\{2\omega_2(t)\}},\\ \omega_2^{\Delta}(t) =&\frac{a_{21}(t)}{a_{22}(t)+a_{23}(t)\exp\{\omega_2(t)\}} - a_{24}(t) - b_2(t)\exp\{\omega_2(t)\},\end{aligned}\right\} \tag{2}$$

where $\omega_i(t)$ are the densities of the i^{th}species at time $t \in \mathbb{T}^+ \big(\mathbb{T}^+$ is a nonempty closed subset of $\mathbb{R}^+ = [0, +\infty)\big)$ and $\omega_i(0) > 0$. ω_i^{Δ} express the delta derivative of the functions $\omega_i(t), i = 1, 2$. $a_{ij}(t), i = 1, 2, j = 1, 2, 3, 4$ and $b_1(t), b_2(t), c(t), d(t)$ are bounded positive almost periodic functions. Clearly, if we set $u_i(t) = \exp\{\omega_i\}, i = 1, 2$ and choose $\mathbb{T}^+ = \mathbb{R}^+$ the system (2) is reduced to the model (1) and $\mathbb{T}^+ = \mathbb{Z}^+$ ($\mathbb{Z}^+$ is the set of nonnegative integer numbers), then the system (2) is reduced to the following discrete system,

$$\begin{aligned}\omega_1(t+1) =&\omega_1(t)\exp\left[\frac{a_{11}(t)}{a_{12}(t)+a_{13}(t)\omega_1(t)} - a_{14}(t) - b_1(t)\omega_1(t) + \frac{c(t)\omega_2(t)}{d(t)+\omega_2^2(t)}\right],\\ \omega_2(t+1) =&\omega_2(t)\exp\left[\frac{a_{21}(t)}{a_{22}(t)+a_{23}(t)\omega_2(t)} - a_{24}(t) - b_2(t)\omega_2(t)\right],\end{aligned}$$

The paper is organized in the following way. In Section 2, we provide some definitions and lemmas which are useful in establishing our main results. In Section 3, we derive sufficient conditions for the permanence of system (2). The sufficient conditions for the existence and uniform asymptotic stability of unique positive almost periodic solution of system (2) are derived in Section 4. In final section, the numeric simulations are given to illustrate the feasibility of the main results.

2. Preliminaries

In this section, we give some definitions and developed lemmas which are useful in the next sections.
As we assumed almost periodic functions on $\mathbb{T}^+$ are bounded, we use the notations

$$f^{\mathcal{L}} = \inf\Big\{f(t) : t \in \mathbb{T}^+\Big\},$$

and

$$f^{\mathcal{U}} = \sup\Big\{f(t) : t \in \mathbb{T}^+\Big\},$$

where $f(t)$ is an almost periodic function. We use the following notations in the paper:

$$\mathscr{A}_1 = \frac{a_{11}^{\mathcal{U}} a_{13}^{\mathcal{U}} e^{\kappa_1}}{\big(a_{12}^{\mathcal{L}} + a_{13}^{\mathcal{L}} e^{\ell_1}\big)^2}, \quad \mathscr{A}_2 = \frac{a_{11}^{\mathcal{L}} a_{13}^{\mathcal{L}} e^{\ell_1}}{\big(a_{12}^{\mathcal{U}} + a_{13}^{\mathcal{U}} e^{\kappa_1}\big)^2},$$

$$\mathscr{B}_1 = \frac{c^{\mathcal{U}}\left(d^{\mathcal{U}} - e^{3\ell_2}\right)}{\left(d^{\mathcal{L}} + e^{2\ell_2}\right)^2}, \quad \mathscr{B}_2 = \frac{c^{\mathcal{L}}\left(d^{\mathcal{L}} - e^{3\kappa_2}\right)}{\left(d^{\mathcal{U}} + e^{2\kappa_2}\right)^2},$$

$$\mathscr{C}_1 = \frac{a_{21}^{\mathcal{U}} a_{23}^{\mathcal{U}} e^{\kappa_2}}{\left(a_{22}^{\mathcal{L}} + a_{23}^{\mathcal{L}} e^{\ell_2}\right)^2}, \quad \mathscr{C}_2 = \frac{a_{21}^{\mathcal{L}} a_{23}^{\mathcal{L}} e^{\ell_2}}{\left(a_{22}^{\mathcal{U}} + a_{23}^{\mathcal{U}} e^{\kappa_2}\right)^2}.$$

Definition 2.1 [5] A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\},$$

$$\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\},$$

and

$$\mu(t) = \rho(t) - t,$$

respectively.

- The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.
- If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$;otherwise $\mathbb{T}^k = \mathbb{T}$.
- A function $g : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 2.2 [5] A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1+\mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Also, we denote the set

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : \mu(t)f(t) > 0, \forall t \in \mathbb{T}\}.$$

Lemma 2.3 *[10] If $a > 0, b > 0$ and $-b \in \mathcal{R}^+$. Then*

$$w^{\Delta}(t) \leq (\geq) a - bw(t), \ w(t) > 0, \ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$w(t) \leq (\geq) \frac{a}{b}\Big[1 + \Big(\frac{bw(t_0)}{a} - 1\Big) e_{(-b)}(t, t_0)\Big], \ t \in [t_0, \infty)_{\mathbb{T}}.$$

Definition 2.4 [13] A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\prod = \{\kappa \in \mathbb{R} : t + \kappa \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.5 [13] Let $\mathbb{T}$ be an almost periodic time scale. Then a function $w \in \mathcal{C}(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ε-translation set of w i.e.,

$$\mathcal{E}\{\varepsilon, w\} = \Big\{\kappa \in \prod : |w(t+\kappa) - w(t)| < \varepsilon, \forall t \in \mathbb{T}\Big\}$$

is a relatively dense set in $\mathbb{T}$ for any positive real number ε.

Definition 2.6 [13] Let $\mathbb{T}$ be a positive almost periodic time scale. Then a function $\phi \in \mathcal{C}(\mathbb{T} \times \mathbb{D}, \mathbb{R}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $w \in \mathbb{D}$ if the ε-translation set of ϕ

$$\mathcal{E}\{\varepsilon, \phi, \mathbb{S}\} = \Big\{\kappa \in \prod : |\phi(t+\kappa) - \phi(t)| < \varepsilon, \forall (t, w) \in \mathbb{T} \times \mathbb{S}\Big\}$$

is a relatively dense set in $\mathbb{T}$ for any positive real number ε, and for each compact subset $\mathbb{S}$ of $\mathbb{D}$.

Next, consider the system

$$w^{\Delta}(t) = \psi(t, w), \tag{3}$$

and its associate product system

$$w^{\Delta}(t) = \psi(t, w), \quad z^{\Delta}(t) = \psi(t, z), \tag{4}$$

where $\psi : \mathbb{T}^+ \times \mathbb{S}_B \to \mathbb{R}^n$, $\mathbb{S}_B = \{w \in \mathbb{R}^n : \|w\| < B\}$, $\psi(t, w)$ is almost periodic in t uniformly for $w \in \mathbb{S}_B$ and is continuous in w.

Lemma 2.7 *[34] Let $\mathcal{V}(t, w, z)$ be Lyapunov function defined on $\mathbb{T}^+ \times \mathbb{S}_B^2$ and satisfies the following conditions*

(i) $\alpha\big(\|w - z\|\big) \le \mathcal{V}(t, w, z) \le \beta\big(\|w - z\|\big)$, *where* $\alpha, \beta \in \mathcal{P}$,

$$\mathcal{P} = \Big\{\gamma \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+) : \gamma(0) = 0 \quad \textit{and} \quad \gamma \quad \textit{is} \quad \textit{increasing}\Big\};$$

(ii) $|\mathcal{V}(t, w, z) - \mathcal{V}(t, w_1, z_1)| \le \mathcal{L}\big(\|w - w_1\| + \|z - z_1\|\big)$, *where* $\mathcal{L} > 0$ *is a constant,*

(iii) $\mathcal{D}^+\mathcal{V}^{\Delta}(t, w, z) \le -\lambda \mathcal{V}(t, w, z)$, *where* $\lambda > 0, -\lambda \in \mathcal{R}^+$.

Further, if there exists a solution $x(t) \in \mathbb{S}$ of system (3) for $t \in \mathbb{T}^+$, where $\mathbb{S} \cup \mathbb{S}_B$ is a compact set, then there exist a unique almost periodic solution $f(t) \in \mathbb{S}$ of system (3), which is uniformly asymptotically stable.

Definition 2.8 System (2) is said to be permanent, if there exist positive constants ℓ, κ such that

$$\ell \le \liminf_{t \to +\infty} \omega_i(t) < \limsup_{t \to +\infty} \omega_i(t) \le \kappa, \; i = 1, 2,$$

for any solution $\big(\omega_1(t), \omega_2(t)\big)$ of (2).

3. Permanence

In this section, we derive the sufficient conditions for the system (2) to be permanent.

Lemma 3.1 *Suppose that*

$$\left.\begin{aligned} a_{11}^{\mathcal{U}} + c^{\mathcal{U}} a_{12}^{\mathcal{L}} &> \big[a_{14}^{\mathcal{L}} + b_1^{\mathcal{L}}\big] a_{12}^{\mathcal{L}} \\ a_{21}^{\mathcal{U}} &> \big[a_{24}^{\mathcal{L}} + b_2^{\mathcal{L}}\big] a_{22}^{\mathcal{L}}. \end{aligned}\right\} \tag{5}$$

Then any positive solution $\big(\omega_1(t), \omega_2(t)\big)$ *of the dynamic system (2) satisfies*

$$\limsup_{t\to+\infty} \omega_1(t) \le \kappa_1 := \frac{1}{b_1^{\mathcal{L}}}\left[\frac{a_{11}^{\mathcal{U}}}{a_{12}^{\mathcal{L}}} - a_{14}^{\mathcal{L}} + c^{\mathcal{U}} - b_1^{\mathcal{L}}\right]$$

and

$$\limsup_{t\to+\infty} \omega_2(t) \le \kappa_2 := \frac{1}{b_2^{\mathcal{L}}}\left[\frac{a_{21}^{\mathcal{U}}}{a_{22}^{\mathcal{L}}} - a_{24}^{\mathcal{L}} - b_2^{\mathcal{L}}\right].$$

Proof It follows from the first equation of the system (2) that

$$\begin{aligned} \omega_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)\exp\{\omega_1(t)\}} - a_{14}(t) - b_1(t)\exp\{\omega_1(t)\} \\ &+ \frac{c(t)\exp\{\omega_2(t)\}}{d(t) + \exp\{2\omega_2(t)\}} \\ \le& \frac{a_{11}(t)}{a_{12}(t)} - a_{14}(t) - b_1(t)\exp\{\omega_1(t)\} + c(t) \\ \le& \frac{a_{11}^{\mathcal{U}}}{a_{12}^{\mathcal{L}}} - a_{14}^{\mathcal{L}} + c^{\mathcal{U}} - b_1^{\mathcal{L}}\exp\{\omega_1(t)\} \\ \le& \frac{a_{11}^{\mathcal{U}}}{a_{12}^{\mathcal{L}}} - a_{14}^{\mathcal{L}} + c^{\mathcal{U}} - b_1^{\mathcal{L}}\big[\omega_1(t) + 1\big]. \end{aligned}$$

By using Lemma 2.3 we have

$$\limsup_{t\to+\infty} \omega_1(t) \le \kappa_1 := \frac{1}{b_1^{\mathcal{L}}}\left[\frac{a_{11}^{\mathcal{U}}}{a_{12}^{\mathcal{L}}} - a_{14}^{\mathcal{L}} + c^{\mathcal{U}} - b_1^{\mathcal{L}}\right].$$

Similarly from the second equation of the system (2) that

$$\begin{aligned} \omega_2^{\Delta}(t) &= \frac{a_{21}(t)}{a_{22}(t) + a_{23}(t)\exp\{\omega_2(t)\}} - a_{24}(t) - b_2(t)\exp\{\omega_2(t)\} \\ &\le \frac{a_{21}(t)}{a_{22}(t)} - a_{24}(t) - b_2(t)\exp\{\omega_2(t)\} \\ &\le \frac{a_{21}^{\mathcal{U}}}{a_{22}^{\mathcal{L}}} - a_{24}^{\mathcal{L}} - b_2^{\mathcal{L}}\big[\omega_2(t) + 1\big]. \end{aligned}$$

From Lemma 2.3, we get

$$\limsup_{t\to+\infty} \omega_2(t) \le \kappa_2 := \frac{1}{b_2^{\mathcal{L}}}\left[\frac{a_{21}^{\mathcal{U}}}{a_{22}^{\mathcal{L}}} - a_{24}^{\mathcal{L}} - b_2^{\mathcal{L}}\right].$$

This completes the proof. ■

Lemma 3.2 *If the inequalities* (5) *and*

$$\left.\begin{aligned} a_{11}^{\mathcal{L}} &> a_{14}^{\mathcal{U}}\big(a_{12}^{\mathcal{U}}+\exp\{\kappa_1\}\big)\\ a_{21}^{\mathcal{L}} &> a_{24}^{\mathcal{U}}\big(a_{22}^{\mathcal{U}}+\exp\{\kappa_2\}\big)\end{aligned}\right\} \tag{6}$$

hold, then any positive solution $\big(\omega_1(t),\omega_2(t)\big)$ *of system (2) satisfies*

$$\liminf_{t\to+\infty}\omega_1(t)\ge \ell_1 := \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\big(a_{12}^{\mathcal{U}}+\exp\{\kappa_1\}\big)}-\frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right],$$

$$\liminf_{t\to+\infty}\omega_2(t)\ge \ell_2 := \ln\left[\frac{a_{21}^{\mathcal{L}}}{b_2^{\mathcal{U}}\big(a_{22}^{\mathcal{U}}+\exp\{\kappa_2\}\big)}-\frac{a_{24}^{\mathcal{U}}}{b_2^{\mathcal{U}}}\right].$$

Proof From Lemma 3.1, we know that

$$\limsup_{t\to+\infty}\omega_1(t)\le\kappa_1,$$

which means that for any $\varepsilon>0$, there exists a $t_0\in\mathbb{T}^+$ such that $\omega_1(t)\le\kappa_1+\varepsilon$ for all $t\ge t_0$. Then for $t\ge t_0$, it follows from the first equation of system (2) that

$$\begin{aligned}\omega_1^{\Delta}(t) =&\frac{a_{11}(t)}{a_{12}(t)+a_{13}(t)\exp\{\omega_1(t)\}}-a_{14}(t)-b_1(t)\exp\{\omega_1(t)\}\\ &\qquad\qquad+\frac{c(t)\exp\{\omega_2(t)\}}{d(t)+\exp\{2\omega_2(t)\}}\\ \ge&\frac{a_{11}^{\mathcal{L}}}{a_{12}^{\mathcal{U}}+\exp\{\kappa_1+\varepsilon\}}-a_{14}^{\mathcal{U}}-b_1^{\mathcal{U}}\exp\{\omega_1(t)\}.\end{aligned}$$

Now we claim that for $t\ge t_0$,

$$\frac{a_{11}^{\mathcal{L}}}{a_{12}^{\mathcal{U}}+\exp\{\kappa_1+\varepsilon\}}-a_{14}^{\mathcal{U}}-b_1^{\mathcal{U}}\exp\{\omega_1(t)\}\le 0. \tag{7}$$

By way of contradiction, assume that there exists a $\hat{t}\ge t_0$ such that

$$\frac{a_{11}^{\mathcal{L}}}{a_{12}^{\mathcal{U}}+\exp\{\kappa_1+\varepsilon\}}-a_{14}^{\mathcal{U}}-b_1^{\mathcal{U}}\exp\{\omega_1(t)\}>0$$

and for any $t\in[t_0,\hat{t})_{\mathbb{T}^+}$,

$$\frac{a_{11}^{\mathcal{L}}}{a_{12}^{\mathcal{U}}+\exp\{\kappa_1+\varepsilon\}}-a_{14}^{\mathcal{U}}-b_1^{\mathcal{U}}\exp\{\omega_1(t)\}\le 0.$$

Then

$$\omega_1(\hat{t})<\ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\big(a_{12}^{\mathcal{U}}+\exp\{\kappa_1+\varepsilon\}\big)}-\frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right]$$

and for any $t \in [t_0, \hat{t})_{\mathbb{T}^+}$,

$$\omega_1(t) \geq \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\left(a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}\right)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right],$$

which implies $\omega_1^{\Delta}(\hat{t}) < 0$. It is contradiction, and hence the inequality in (7) holds for all $t \geq t_0$, and

$$\omega_1(t) \geq \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\left(a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}\right)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right],$$

consequently

$$\liminf_{t\to+\infty} \omega_1(t) \geq \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\left(a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}\right)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right].$$

Since ε is arbitrary small and from the first inequality in (6), we have

$$\liminf_{t\to+\infty} \omega_1(t) \geq \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\left(a_{12}^{\mathcal{U}} + \exp\{\kappa_1\}\right)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right].$$

Analogously, by the second inequality in (6), we obtain that

$$\liminf_{t\to+\infty} \omega_2(t) \geq \ln\left[\frac{a_{21}^{\mathcal{L}}}{b_2^{\mathcal{U}}\left(a_{22}^{\mathcal{U}} + \exp\{\kappa_2\}\right)} - \frac{a_{24}^{\mathcal{U}}}{b_2^{\mathcal{U}}}\right].$$

This completes the proof. ■

Theorem 3.3 *Under the assumptions* (5) *and* (6)*, the system* (2) *is permanent.*

Proof From Lemmas 3.1 and 3.2, the system (2) is permanent. ■

4. Positive almost periodic solution

In this section, we establish sufficient conditions for the existence, uniqueness and uniform asymptotic stability of positive almost periodic solution of system (2). Define

$$\Lambda = \Big\{\big(\omega_1(t), \omega_2(t)\big) : \big(\omega_1(t), \omega_2(t)\big) \text{ is a solution of } (2) \\ \text{and } 0 < \ell_i \leq \omega_i(t) \leq \kappa_i,\ i = 1, 2\Big\}.$$

It is clear that Λ is invariant set of system (2).

Theorem 4.1 *Suppose that* (5) *and* (6) *are satisfied, then* $\Lambda \neq \varnothing$.

Proof The almost periodicity of $a_{ij}(t)$, $i = 1, 2, 3, 4; j = 1, 2$ implies that there is a sequence $\{\theta_k\} \subseteq \mathbb{T}^+$ with $\theta_k \to +\infty$ such that

$$a_{ij}(t + \theta_k) \to a_{ij}(t),\ \ as\ k \to +\infty,\ i = 1, 2, 3, 4; j = 1, 2.$$

From Lemma 3.1 and 3.2, for each sufficiently small $\epsilon > 0$, there exists a $\tau \in \mathbb{T}^+$ such that

$$\ell_i - \epsilon \leq \omega_i(t) \leq \kappa_i + \epsilon, \ \textit{for all}\ t \geq \tau,\ i = 1, 2.$$

Set $\omega_{ik}(t) = \omega_i(t + \theta_k)$ for $t \geq \tau - \theta_k$, $k = 1, 2, \cdots$. For any positive integer m, there exists a sequence $\{\omega_{ik}(t) : k \geq m\}$ such that the sequence $\{\omega_{ik}(t)\}$ has a subsequence, denoted by $\{\omega^*_{ik}(t)\}(\omega^*_{ik}(t) = \omega_i(t + \theta^*_k))$, converging on any finite interval of $\mathbb{T}^+$ as $k \to +\infty$. So we have a sequence $\{w_i(t)\}$ such that for $t \in \mathbb{T}^+$,

$$\omega^*_{ik}(t) \to w_k(t),\ \textit{as}\ k \to +\infty,\ i = 1, 2. \tag{8}$$

It is easy to see that the above sequence $\{\theta^*_k\} \subseteq \mathbb{T}^+$ with $\theta^*_k \to +\infty$ for $k \to +\infty$ such that

$$a_{ij}(t + \theta^*_k) \to a_{ij}(t),\ \textit{as}\ k \to +\infty,\ i = 1, 2, 3, 4; j = 1, 2.$$

Which, together with (8) and

$$\begin{aligned}
\omega_1^{*\Delta}(t) =& \frac{a_{11}(t+\theta^*_k)}{a_{12}(t+\theta^*_k) + a_{13}(t+\theta^*_k)\exp\{\omega_1(t)\}} - a_{14}(t+\theta^*_k) - b_1(t+\theta^*_k)\exp\{\omega_1(t)\} \\
&+ \frac{c(t+\theta^*_k)\exp\{\omega_2(t)\}}{d(t+\theta^*_k) + \exp\{2\omega_2(t)\}}, \\
\omega_2^{*\Delta}(t) =& \frac{a_{21}(t+\theta^*_k)}{a_{22}(t+\theta^*_k) + a_{23}(t+\theta^*_k)\exp\{\omega_2(t)\}} - a_{24}(t+\theta^*_k) - b_2(t+\theta^*_k)\exp\{\omega_2(t)\},
\end{aligned}$$

yields

$$\begin{aligned}
w_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)\exp\{w_1(t)\}} - a_{14}(t) - b_1(t)\exp\{w_1(t)\} \\
&+ \frac{c(t)\exp\{w_2(t)\}}{d(t) + \exp\{2w_2(t)\}}, \\
w_2^{\Delta}(t) =& \frac{a_{21}(t)}{a_{22}(t) + a_{23}(t)\exp\{w_2(t)\}} - a_{24}(t) - b_2(t)\exp\{w_2(t)\},
\end{aligned}$$

It is clear that $\big(w_1(t), w_2(t)\big)$ is a solution of system (2) and

$$\ell_i - \epsilon \leq w_i(t) \leq \kappa_i + \epsilon,\ \textit{for}\ t \in \mathbb{T}^+,\ i = 1, 2.$$

Since ϵ was arbitrary, it follows that

$$\ell_i \leq w_i(t) \leq \kappa_i,\ \textit{for}\ t \in \mathbb{T}^+,\ i = 1, 2.$$

This completes the proof. ∎

Theorem 4.2 *Assume that* (5), (6), $\Gamma_1 > 0$ *and* $\Gamma_2 > 0$, *where*

$$\Gamma_1 = \Big[\big(2b_1^{\mathcal{L}}e^{\ell_1} + 2b_1^{\mathcal{L}}\mathscr{A}_2e^{\ell_1} + \mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2\big) - \big(2\mathscr{A}_1 + \mu^{\mathcal{U}}\left(b_1^{\mathcal{U}}\right)^2 e^{2\kappa_1} + \mu^{\mathcal{U}}\mathscr{A}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1\big)\Big],$$

$$\Gamma_2 = \Big[\big(\mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2 + 2b_2^{\mathcal{L}}e^{\ell_2}\big(1 + \mu^{\mathcal{L}}\mathscr{C}_2\big)\big) - \big(\mu^{\mathcal{U}}\mathscr{B}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1 + 2\mathscr{C}_1 + \mu^{\mathcal{U}}\mathscr{C}_1^2 + \mu^{\mathcal{U}}(b_2^{\mathcal{U}})^2e^{2\kappa_2}\big)\Big],$$

are satisfied. Then the dynamic system (2) *has a unique almost periodic solution* $\big(\omega_1(t), \omega_2(t)\big) \in \Lambda$ *and is uniformly asymptotically stable.*

Proof From Theorem 4.1 that there exists a solution $\big(\omega_1(t), \omega_2(t)\big)$ of system (2) such that

$$\ell_i \leq \omega_i(t) \leq \kappa_i,$$

for $t \in \mathbb{T}^+$, $i = 1, 2$.
Define

$$\|\big(\omega_1(t), \omega_2(t)\big)\| = |\omega_1(t)| + |\omega_2(t)|, \quad \big(\omega_1(t), \omega_2(t)\big) \in \mathbb{R}_+^2.$$

Assume that $\mathcal{W}_1(t) = \big(\omega_1(t), \omega_2(t)\big)$, $\mathcal{W}_2(t) = \big(w_1(t), w_2(t)\big)$ are any two positive solutions of system (2), then

$$\|\mathcal{W}_1\| \leq \kappa_1 + \kappa_2$$

and

$$\|\mathcal{W}_2\| \leq \kappa_1 + \kappa_2.$$

We consider the associate product system of system (2) as follows

$$\left.\begin{aligned}
\omega_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)\exp\{\omega_1(t)\}} - a_{14}(t) - b_1(t)\exp\{\omega_1(t)\} \\
& + \frac{c(t)\exp\{\omega_2(t)\}}{d(t) + \exp\{2\omega_2(t)\}}, \\
\omega_2^{\Delta}(t) =& \frac{a_{21}(t)}{a_{22}(t) + a_{23}(t)\exp\{\omega_2(t)\}} - a_{24}(t) - b_2(t)\exp\{\omega_2(t)\}, \\
w_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)\exp\{w_1(t)\}} - a_{14}(t) - b_1(t)\exp\{w_1(t)\} \\
& + \frac{c(t)\exp\{w_2(t)\}}{d(t) + \exp\{2w_2(t)\}}, \\
w_2^{\Delta}(t) =& \frac{a_{21}(t)}{a_{22}(t) + a_{23}(t)\exp\{w_2(t)\}} - a_{24}(t) - b_2(t)\exp\{w_2(t)\}.
\end{aligned}\right\} \quad (9)$$

Construct the following Lyapunov function $\mathcal{V}\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big)$ on $\mathbb{T}^+ \times \Omega \times \Omega$ by

$$\mathcal{V}\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) = \big(\omega_1(t) - w_1(t)\big)^2 + \big(\omega_2(t) - w_2(t)\big)^2.$$

It is obvious that the norm

$$\|\mathcal{W}_1(t) - \mathcal{W}_2(t)\| = |\omega_1(t) - w_1(t)| + |\omega_2(t) - w_2(t)|$$

is equivalent to

$$\|\mathcal{W}_1(t) - \mathcal{W}_2(t)\|_* = \big[\big(\omega_1(t) - w_1(t)\big)^2 + \big(\omega_2(t) - w_2(t)\big)^2\big]^{\frac{1}{2}},$$

in other words, there exist two constants $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\delta_1\|\mathcal{W}_1(t) - \mathcal{W}_2(t)\| \leq \|\mathcal{W}_1(t) - \mathcal{W}_2(t)\|_* \leq \delta_2\|\mathcal{W}_1(t) - \mathcal{W}_2(t)\|,$$

and hence we have

$$\big(\delta_1\|\mathcal{W}_1(t) - \mathcal{W}_2(t)\|\big)^2 \leq \mathcal{V}\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) \leq \big(\delta_2\|\mathcal{W}_1(t) - \mathcal{W}_2(t)\|\big)^2.$$

Let $\alpha, \beta \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$, $\alpha(\omega) = \delta_1^2\omega^2$, $\beta(\omega) = \delta_2^2\omega^2$, then the assumption (i) of Lemma 2.7 is satisfied. On the other hand, we have

$$\begin{aligned}
&\Big|\mathcal{V}\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) - \mathcal{V}\big(t, \mathcal{W}_1^*(t), \mathcal{W}_2^*(t)\big)\Big| \\
&= \Big|\big(\omega_1(t) - w_1(t)\big)^2 + \big(\omega_2(t) - w_2(t)\big)^2 - \big(\omega_1^*(t) - w_1^*(t)\big)^2 - \big(\omega_2^*(t) - w_2^*(t)\big)^2\Big| \\
&\leq \Big|\big(\omega_1(t) - w_1(t)\big) - \big(\omega_1^*(t) - w_1^*(t)\big)\Big|\Big|\big(\omega_1(t) - w_1(t)\big) + \big(\omega_1^*(t) - w_1^*(t)\big)\Big| \\
&\quad \Big|\big(\omega_2(t) - w_2(t)\big) - \big(\omega_2^*(t) - w_2^*(t)\big)\Big|\Big|\big(\omega_2(t) - w_2(t)\big) + \big(\omega_2^*(t) - w_2^*(t)\big)\Big| \\
&\leq \Big|\big(\omega_1(t) - w_1(t)\big) - \big(\omega_1^*(t) - w_1^*(t)\big)\Big|\Big(|\omega_1(t)| + |w_1(t)| + |\omega_1^*(t)| + |w_1^*(t)|\Big) \\
&\quad \Big|\big(\omega_2(t) - w_2(t)\big) - \big(\omega_2^*(t) - w_2^*(t)\big)\Big|\Big(|\omega_2(t)| + |w_2(t)| + |\omega_2^*(t)| + |w_2^*(t)|\Big) \\
&\leq \mathcal{L}\Big(|\omega_1(t) - \omega_1^*(t)| + |\omega_2(t) - \omega_2^*(t)| + |w_1(t) - w_1^*(t)| + |w_2(t) - w_2^*(t)|\Big) \\
&= \mathcal{L}\Big(\|\mathcal{W}_1(t) - \mathcal{W}_1^*(t)\| + \|\mathcal{W}_2(t) - \mathcal{W}_2^*(t)\|\Big),
\end{aligned}$$

where $\mathcal{W}_1^*(t) = \big(\omega_1^*, \omega_2^*\big), \mathcal{W}_2^*(t) = \big(w_1^*, w_2^*\big)$, and $\mathcal{L} = 4\max\{\kappa_i, i = 1, 2\}$. Hence, the assumption (ii) of Lemma 2.7 is satisfied.

Now, estimating the right derivative $\mathcal{D}^+\mathcal{V}^\Delta$ of $\mathcal{V}$ along with associate product system (9), we obtain

$$\begin{aligned}
&\mathcal{D}^+\mathcal{V}^\Delta\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)\\
&\quad= \big(\omega_1(t)-w_1(t)\big)^\Delta\big(\omega_1(t)-w_1(t)\big)+[\omega_1(\sigma(t))-w_1(\sigma(t))]\big(\omega_1(t)-w_1(t)\big)\\
&\quad+\big(\omega_2(t)-w_2(t)\big)^\Delta\big(\omega_2(t)-w_2(t)\big)+[\omega_2(\sigma(t))-w_2(\sigma(t))]\big(\omega_2(t)-w_2(t)\big)\\
&\quad= \big(\omega_1(t)-w_1(t)\big)^\Delta\big(\omega_1(t)-w_1(t)\big)+\big[\big(\mu(t)\omega_1^\Delta(t)+\omega_1(t)\big)\\
&\qquad\qquad-\big(\mu(t)w_1^\Delta(t)+w_1(t)\big)\big]\big(\omega_1(t)-w_1(t)\big)^\Delta\\
&\qquad+\big(\omega_2(t)-w_2(t)\big)^\Delta\big(\omega_2(t)-w_2(t)\big)+\big[\big(\mu(t)\omega_2^\Delta(t)+\omega_2(t)\big)\\
&\qquad\qquad-\big(\mu(t)w_2^\Delta(t)+w_2(t)\big)\big]\big(\omega_2(t)-w_2(t)\big)^\Delta\\
&\quad=\Big[2\big(\omega_1(t)-w_1(t)\big)+\mu(t)\big(\omega_1(t)-w_1(t)\big)^\Delta\Big]\big(\omega_1(t)-w_1(t)\big)^\Delta\\
&\qquad+\Big[2\big(\omega_2(t)-w_2(t)\big)+\mu(t)\big(\omega_2(t)-w_2(t)\big)^\Delta\Big]\big(\omega_2(t)-w_2(t)\big)^\Delta.
\end{aligned}$$

So,

$$\mathcal{D}^+\mathcal{V}^\Delta\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)=\mathcal{V}_1+\mathcal{V}_2, \tag{10}$$

where

$$\begin{aligned}
\mathcal{V}_1&=\Big[2\big(\omega_1(t)-w_1(t)\big)+\mu(t)\big(\omega_1(t)-w_1(t)\big)^\Delta\Big]\big(\omega_1(t)-w_1(t)\big)^\Delta,\\
\mathcal{V}_2&=\Big[2\big(\omega_2(t)-w_2(t)\big)+\mu(t)\big(\omega_2(t)-w_2(t)\big)^\Delta\Big]\big(\omega_2(t)-w_2(t)\big)^\Delta.
\end{aligned}$$

From the system (9), we have

$$\begin{aligned}
&\big(\omega_1(t)-w_1(t)\big)^\Delta=a_{11}(t)\left[\frac{1}{a_{12}(t)+a_{13}(t)\exp\{\omega_1(t)\}}-\frac{1}{a_{12}(t)+a_{13}(t)\exp\{w_1(t)\}}\right]\\
&-b_1(t)[\exp\{\omega_1(t)\}-\exp\{w_1(t)\}]+c(t)\left[\frac{\exp\{\omega_2(t)\}}{d(t)+\exp\{2\omega_2(t)\}}-\frac{\exp\{w_2(t)\}}{d(t)+\exp\{2w_2(t)\}}\right]
\end{aligned}$$

and

$$\begin{aligned}
\big(\omega_2(t)-w_2(t)\big)^\Delta=a_{21}(t)&\left[\frac{1}{a_{22}(t)+a_{23}(t)\exp\{\omega_2(t)\}}-\frac{1}{a_{22}(t)+a_{23}(t)\exp\{w_2(t)\}}\right]\\
&-b_2(t)[\exp\{\omega_2(t)\}-\exp\{w_2(t)\}].
\end{aligned}$$

By mean value theorem, there exit $\xi_i(t),\eta_i(t),\ i=1,2$ lie between $\omega_i(t)$ and $w_i(t)$, and $\xi(t)$ lie between $\omega_2(t)$ and $w_2(t)$ such that

$$\exp\{\omega_i(t)\}-\exp\{w_i(t)\}=\exp\{\xi_i(t)\}[\omega_i(t)-w_i(t)],$$

$$\frac{\exp\{\omega_2(t)\}}{d(t)+\exp\{2\omega_2(t)\}}-\frac{\exp\{w_2(t)\}}{d(t)+\exp\{2w_2(t)\}}=\left[\frac{d-\exp\{3\xi(t)\}}{(d+\exp\{2\xi(t)\})^2}\right][\omega_2(t)-w_2(t)],$$

$$\frac{1}{a_{i2}(t)+a_{i3}(t)\exp\{\omega_i(t)\}}-\frac{1}{a_{i2}(t)+a_{i3}(t)\exp\{w_i(t)\}} = \left[\frac{a_{i3}(t)\exp\{\eta_i(t)\}}{(a_{i2}(t)+a_{i3}(t)\exp\{\eta_i(t)\})^2}\right][\omega_i(t)-w_i(t)].$$

Therefore,

$$\big(\omega_1(t)-w_1(t)\big)^{\Delta} = \left[\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right][\omega_1(t)-w_1(t)] - b_1(t)\exp\{\xi_1(t)\}[\omega_1(t)-w_1(t)] + \left[\frac{c(t)(d-\exp\{3\xi(t)\})}{(d+\exp\{2\xi(t)\})^2}\right][\omega_2(t)-w_2(t)],$$

and

$$\big(\omega_2(t)-w_2(t)\big)^{\Delta} = \left[\frac{a_{21}(t)a_{23}(t)\exp\{\eta_2(t)\}}{(a_{22}(t)+a_{23}(t)\exp\{\eta_2(t)\})^2}\right][\omega_2(t)-w_2(t)] - b_2(t)\exp\{\xi_2(t)\}[\omega_2(t)-w_2(t)].$$

Now from (10), we have

$$\begin{aligned}
\mathcal{V}_1 =& \Bigg[2\big(\omega_1(t)-w_1(t)\big)+\mu(t)\Bigg(\left[\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right][\omega_1(t)-w_1(t)] \\
&- b_1(t)\exp\{\xi_1(t)\}[\omega_1(t)-w_1(t)] + \left[\frac{c(t)(d-\exp\{3\xi(t)\})}{(d+\exp\{2\xi(t)\})^2}\right][\omega_2(t)-w_2(t)]\Bigg)\Bigg] \\
&\times \Bigg[\left[\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right][\omega_1(t)-w_1(t)] \\
&- b_1(t)\exp\{\xi_1(t)\}[\omega_1(t)-w_1(t)] + \left[\frac{c(t)(d-\exp\{3\xi(t)\})}{(d+\exp\{2\xi(t)\})^2}\right][\omega_2(t)-w_2(t)]\Bigg] \\
=& \Bigg[2\left(\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2} - b_1(t)\exp\{\xi_1(t)\}\right) \\
&+ \mu(t)(b_1(t))^2\exp\{2\xi_1(t)\} + \mu(t)\left(\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right)^2 \\
&- \frac{2b_1(t)a_{11}(t)a_{13}(t)\exp\{\xi_1(t)+\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\Bigg][\omega_1(t)-w_1(t)]^2 \\
&+ \mu(t)\left[\frac{c(t)(d(t)-\exp\{3\xi(t)\})}{(d(t)+\exp\{2\xi(t)\})^2}\right]^2[\omega_2(t)-w_2(t)]^2 \\
&+ 2\Bigg[\mu(t)\left(\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right)\left(\frac{c(t)(d(t)-\exp\{3\xi(t)\})}{(d(t)+\exp\{2\xi(t)\})^2}\right) \\
&+ \big(1-\mu(t)b_1(t)\exp\{\xi_1(t)\}\big)\left(\frac{c(t)(d(t)-\exp\{3\xi(t)\})}{(d(t)+\exp\{2\xi(t)\})^2}\right)\Bigg] \\
&\times [\omega_1(t)-w_1(t)][\omega_2(t)-w_2(t)]
\end{aligned}$$

$$\leq \Big[2\mathscr{A}_1 - 2b_1^{\mathcal{L}}e^{\ell_1} + \mu^{\mathcal{U}}\left(b_1^{\mathcal{U}}\right)^2 e^{2\kappa_1} + \mu^{\mathcal{U}}\mathscr{A}_1^2 - 2b_1^{\mathcal{L}}\mathscr{A}_2 e^{\ell_1}\Big][\omega_1(t) - w_1(t)]^2$$
$$+ \mu^{\mathcal{U}}\mathscr{B}_1^2[\omega_2(t) - w_2(t)]^2$$
$$+ 2[\mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1 - \mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2][\omega_1(t) - w_1(t)][\omega_2(t) - w_2(t)]$$

Since $2ab \leq a^2 + b^2$ for any $a, b \in \mathbb{R}$, it follows that

$$\left.\begin{aligned} \mathcal{V}_1 \leq -\Big[&\big(2b_1^{\mathcal{L}}e^{\ell_1} + 2b_1^{\mathcal{L}}\mathscr{A}_2 e^{\ell_1} + \mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2\big) \\ &- \big(2\mathscr{A}_1 + \mu^{\mathcal{U}}\left(b_1^{\mathcal{U}}\right)^2 e^{2\kappa_1} + \mu^{\mathcal{U}}\mathscr{A}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1\big)\Big][\omega_1(t) - w_1(t)]^2 \\ &+ \Big[\mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2 - \big(\mu^{\mathcal{U}}\mathscr{B}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1\big)\Big][\omega_2(t) - w_2(t)]^2. \end{aligned}\right\} \tag{11}$$

Similarly, we can find

$$\mathcal{V}_2 \leq -\Big[2b_2^{\mathcal{L}}e^{\ell_2}\big(1 + \mu^{\mathcal{L}}\mathscr{C}_2\big) - \big(2\mathscr{C}_1 + \mu^{\mathcal{U}}\mathscr{C}_1^2 + \mu^{\mathcal{U}}(b_2^{\mathcal{U}})^2 e^{2\kappa_2}\big)\Big][\omega_2(t) - w_2(t)]^2. \tag{12}$$

From (10), (11) and (12), we get

$$\begin{aligned} &D^+\mathcal{V}^{\Delta}\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) = \mathcal{V}_1 + \mathcal{V}_2 \\ &= -\Big[\big(2b_1^{\mathcal{L}}e^{\ell_1} + 2b_1^{\mathcal{L}}\mathscr{A}_2 e^{\ell_1} + \mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2\big) \\ &\qquad - \big(2\mathscr{A}_1 + \mu^{\mathcal{U}}\left(b_1^{\mathcal{U}}\right)^2 e^{2\kappa_1} + \mu^{\mathcal{U}}\mathscr{A}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1\big)\Big][\omega_1(t) - w_1(t)]^2 \\ &\quad - \Big[\big(\mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2 + 2b_2^{\mathcal{L}}e^{\ell_2}\big(1 + \mu^{\mathcal{L}}\mathscr{C}_2\big)\big) \\ &\quad - \big(\mu^{\mathcal{U}}\mathscr{B}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1 + 2\mathscr{C}_1 + \mu^{\mathcal{U}}\mathscr{C}_1^2 + \mu^{\mathcal{U}}(b_2^{\mathcal{U}})^2 e^{2\kappa_2}\big)\Big][\omega_2(t) - w_2(t)]^2 \\ &= -\Gamma_1[\omega_1(t) - w_1(t)]^2 - \Gamma_2[\omega_2(t) - w_2(t)]^2 \\ &\leq -\lambda\mathcal{V}(t, \mathcal{W}_1(t), \mathcal{W}_2(t)). \end{aligned}$$

where $\lambda = \min\{\Gamma_i : i = 1, 2\} > 0$ and $-\lambda \in \mathcal{R}^+$. Thus, the assumption (iii) of Lemma 2.7 is satisfied and hence, it follows from Lemma 2.7 that there exists a unique uniformly asymptotically stable almost periodic solution $\big(\omega_1(t), \omega_2(t)\big)$ of dynamic system (2) and $\big(\omega_1(t), \omega_2(t)\big) \in \Lambda$. This completes the proof. ∎

5. Numerical simulations

In this section we present an example to check the validity of our main results.

Example 5.1 Consider the following system for $\mathbb{T}^+ = \mathbb{R}^+$.

$$\left.\begin{aligned} u_1'(t) &= u_1(t)\left[\frac{a_{11}(t)}{a_{12}(t)+a_{13}(t)u_1(t)} - a_{14}(t) - b_1(t)u_1(t) + \frac{c(t)u_2(t)}{d(t)+u_2^2(t)}\right], \\ u_2'(t) &= u_2(t)\left[\frac{a_{21}(t)}{a_{22}(t)+a_{23}(t)u_2(t)} - a_{24}(t) - b_2(t)u_2(t)\right], \end{aligned}\right\} \tag{13}$$

where

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \end{bmatrix} = \begin{bmatrix} 50+0.1\sin(\sqrt{3}t) & 48+0.1\sin(\sqrt{5}t) \\ 15+0.2\sin(\sqrt{2}t) & 28+0.1\sin(\sqrt{3}t) \\ 0.2+0.1\sin(\sqrt{5}t) & 120+0.2\sin(\sqrt{2}t) \\ 0.03+0.01\sin(\sqrt{2}t) & 0.002+0.01\sin(\sqrt{3}t) \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1.4+0.1\cos(\sqrt{2}t) \\ 1.4-0.1\sin(\sqrt{5}t) \end{bmatrix}, \quad \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0.4+0.1\sin(\sqrt{2}t) \\ 3.2+0.1\sin(\sqrt{3}t) \end{bmatrix}.$$

By calculating, we get

$$\begin{aligned} 57.5 = a_{11}^{\mathcal{U}} + c^{\mathcal{U}} a_{12}^{\mathcal{L}} > 19.536 = \left[a_{14}^{\mathcal{L}} + b_1^{\mathcal{L}}\right] a_{12}^{\mathcal{L}}, \\ 48.1 = a_{21}^{\mathcal{U}} > 36.0468 = \left[a_{24}^{\mathcal{L}} + b_2^{\mathcal{L}}\right] a_{22}^{\mathcal{L}}, \end{aligned}$$

which shows that (5) holds and $\kappa_1 = 1.973180873, \kappa_2 = 0.3323187208$. Now we check (6),

$$\begin{aligned} 49.9 = a_{11}^{\mathcal{L}} > 0.8957408724 = a_{14}^{\mathcal{U}}\left(a_{12}^{\mathcal{U}} + \exp\{\kappa_1\}\right), \\ 47.9 = a_{21}^{\mathcal{L}} > 0.3539303657 = a_{24}^{\mathcal{U}}\left(a_{22}^{\mathcal{U}} + \exp\{\kappa_2\}\right). \end{aligned}$$

So, $\ell_1 = 0.3776703951, \ell_2 = 0.07204048280$. From these values we obtain,

$$\begin{aligned} &\mathscr{A}_1 = 0.4840130676, \ \mathscr{A}_2 = 0.02416120093, \ \mathscr{B}_1 = 0.05685627445, \\ &\mathscr{B}_2 = 0.04254742499, \ \mathscr{C}_1 = 0.3284875457, \ \mathscr{C}_2 = 0.1610553506. \end{aligned}$$

By above values (note that for $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$), we get

$$\Gamma_1 = 2.859856503, \quad \Gamma_2 = 2.080385645.$$

$\lambda = \min\{\Gamma_i : i = 1, 2\} > 0$ and $-\lambda \in \mathcal{R}^+$. From Fig. 1-3, it is easy to see that for system (13) there exists a positive almost periodic solution denoted by $\big(\omega_1^*(t), \omega_2^*(t)\big)$. Moreover, Fig. 4-5 shows that any positive solution $\big(\omega_1(t), \omega_2(t)\big)$ tends to the above almost periodic solution $\big(\omega_1^*(t), \omega_2^*(t)\big)$.

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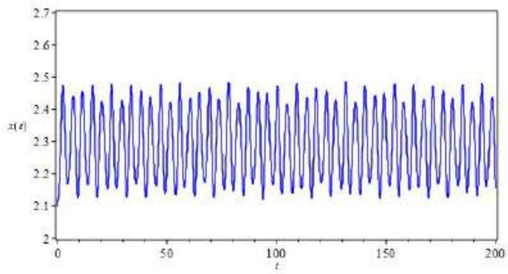

Figure 1. Positive almost periodic solution of system (13). Time series of $u_1^*(t)$ with initial value $u_1^*(0) = 2.1$ and t over $[0, 300]$.

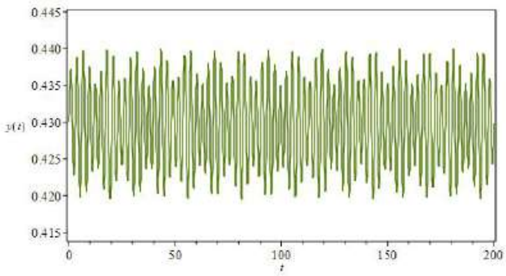

Figure 2. Positive almost periodic solution of system (13). Time series of $x_2^*(t)$ with initial value $x_1^*(0) = 0.43$ and t over $[0, 300]$.

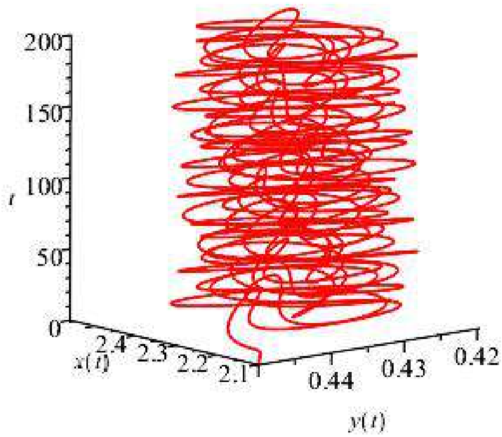

Figure 3. Positive almost periodic solution of system (13). 3-dimensional phase portrait of $u_1^*(t)$ and $u_2^*(t)$ with initial values $(2.1, 0.45)$ for $t \in [0, 200]$.

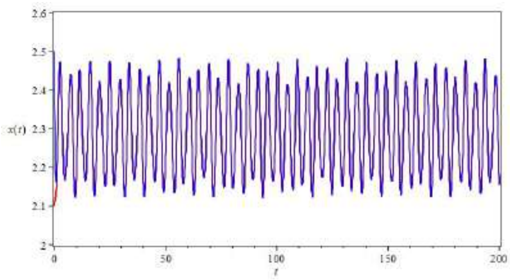

Figure 4. Uniformly asymptotic stability of system (13). Time series of $u_1(t)$ and $u_1^*(t)$ with initial values $u_1(0) = 2.1$, $u_1^*(0) = 2.5$ and t over $[0, 200]$.

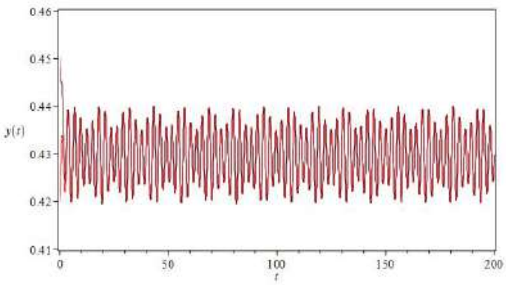

Figure 5. Uniformly asymptotic stability of system (13). Time series of $u_2(t)$ and $u_2^*(t)$ with initial values $u_2(0) = 0.42$, $u_2^*(0) = 0.45$ and t over $[0, 200]$.

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Almost periodic positive solutions for a time-delayed SIR epidemic model with saturated treatment on time scales

Kapula Rajendra Prasad[†], **Mahammad Khuddush**[†*], **Kuparala Venkata Vidyasagar**[†‡]

[†] *College of Science and Technology, Department of Applied Mathematics, Andhra University, Visakhapatnam, India-530003*

[‡] *Department of Mathematics, Government Degree College for Women, Marripalem, Koyyuru Mandal, Visakhapatnam, India-531116*

Email(s): rajendra92@rediffmail.com, khuddush89@gmail.com, vidyavijaya08@gmail.com

Abstract. In this paper, we study a non-autonomous time-delayed SIR epidemic model which involves almost periodic incidence rate and saturated treatment function on time scales. By utilizing some dynamic inequalities on time scales, sufficient conditions are derived for the permanence of the SIR epidemic model and we also obtain the existence and uniform asymptotic stability of almost periodic positive solutions for the addressed SIR model by Lyapunov functional method. Finally numerical simulations are given to demonstrate our theoretical results.

Keywords: SIR model, time scale, almost periodic incidence rate, almost periodic positive solution, permanence, uniform asymptotic stability.
AMS Subject Classification 2010: 92D30, 34N05.

1 Introduction

Infectious diseases are caused by pathogenic microorganisms, such as bacteria, viruses, fungi and parasites. The diseases can spread directly or indirectly from one person to another or from birds or animals to humans and these diseases are a leading cause of death. Despite all the advancement in medicines, infectious disease outbreaks still constitute a significant threat to the public health and economy. Mathematical modeling has become a valuable tool to understand the dynamics of infectious disease and to support the development of control strategies and studied by many researchers [11, 16, 17] and references therein.

The differential, difference and dynamic equations on time scales are three equations play important role for modelling in the environment.Among them, the theory of dynamic equations

*Corresponding author.
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on time scales is the most recent and was introduced by Hilger in his PhD thesis in 1988 [7] with three main features: unification, extension and discretization. Since a time scale is any closed and nonempty subset of the real numbers set. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours Hence, dynamic equations on a time scale have a potential for applications. In the population dynamics, the insect population can be better modelled using time scale calculus. The reason is that they evolve continuously while in season, die out in winter while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a non-overlapping population. Some of the good contributions in this field can be found in [1–3, 14, 15].

In 2016, Bohner and Streipert [4] considered the SIS model,

$$\begin{aligned} S^{\Delta}(t) &= I(t)\Big[-\beta S(\sigma(t)) + \gamma\Big], \quad S(t) > 0, \\ I^{\Delta}(t) &= I(t)\Big[\beta S(\sigma(t)) - \gamma\Big], \quad I(t) \geq 0, \end{aligned}$$

where $\beta > 0, \gamma > 0$ are the transmission and recovery rates of the disease and $\sigma(t)$ denotes the forward jump operator and discussed the stability of the steady states of the model. In [5], Bohner, Streipert and Torres derived exact solution of non-autonomous SIR epidemic model,

$$\begin{aligned} x^{\Delta}(t) &= -\frac{b(t)x(t)y(\sigma(t))}{x(t)+y(t)} \\ y^{\Delta}(t) &= \frac{b(t)x(t)y(\sigma(t))}{x(t)+y(t)} - c(t)y(\sigma(t)) \\ z^{\Delta}(t) &= c(t)z(\sigma(t)), \quad x(t), y(t) > 0, \end{aligned}$$

and then analyzed the stability of the solutions to corresponding autonomous model. In the real world phenomena, since the almost periodic variation of the environment plays a crucial role in many biological and ecological dynamical systems and is more frequent and general than the periodic variation of the environment. The concept of almost periodic time scales was proposed by Li and Wang [10]. Based on this concept, some works have been done [12–15].

Recently, Bohner and Streipert [6] analysed the existence and globally asymptotic stability of a ω-periodic solution to the discrete SIS model,

$$\begin{aligned} \Delta S_t &= -\beta_t S_{t+1} I_t + \gamma_t I_t, \\ \Delta I_t &= \beta_t S_{t+1} I_t - \gamma_t I_t. \end{aligned}$$

Motivated by aforementioned works and mainly [9, 18], in this paper we study the following time-delayed SIR model on time scales.

2 Model description and preliminaries

In this section, we consider an SIR model with saturated and periodic incidence rate and saturated treatment function, whose corresponding continuous ($\mathbb{R}$) model has been studied in [9, 18].

The population is divided into three classes: the susceptible class S, the infectious class I, and the recovered class R. The transition dynamics associated with these subpopulations are illustrated in Fig. 1.

Based on the above discussion, we make the following assumptions:

(1) The infection is transmitted to humans by a vector, i.e., susceptible persons receive the infection from infectious vectors, and susceptible vectors receive the infection from infectious persons.

(2) When a susceptible vector is infected by a human, there is a fixed time τ during which the infectious agents develop in the vector, and it is after that time that the infected vector can infect the susceptible human population.

(3) The number of newly infected individuals per time unit is proportional to $S(t)\mathtt{u}(t)/\big(1 + a(t)\mathtt{u}(t)\big)$, where $\mathtt{u}(t)$ the number of infectious vectors in the community at time t, and $(1 + a(t)\mathtt{u}(t))^{-1}$ represents the saturation effect when the population of infectious vectors is large.

(4) The total vector population is very large and $\mathtt{u}(t)$ is proportional to $I(t-\tau)$.

Using above assumptions, we propose the delayed susceptible-infected-recovered (SIR) model with saturated treatment on time scales by

$$\left.\begin{aligned} S^{\Delta}(t) &= A(t) - \alpha(t)S(t) - \frac{\chi(t)S(t)I(t-\tau)}{1+a(t)I(t-\tau)},\\ I^{\Delta}(t) &= \frac{\chi(t)S(t)I(t-\tau)}{1+a(t)I(t-\tau)} - [\alpha(t)+\beta(t)+\gamma(t)]I(t) - \frac{b(t)I(t)}{1+c(t)I(t)},\\ R^{\Delta}(t) &= \gamma(t)I(t) + \frac{b(t)I(t)}{1+c(t)I(t)} - \alpha(t)R(t), \end{aligned}\right\} \tag{1}$$

where $t \in \mathbb{T}$(time scale). Motivated by biological realism, we take the contact rate as $\chi(t) = d+\delta\sin(\pi/6)t$, (for more details refer [9]) and all other parameters are positive. While contacting with infected individuals, the susceptible individuals become infected at a saturated incidence rate $\frac{\chi SI}{1+aI}$. Through treatment, the infected individuals recover at a saturated treatment function $\frac{bI}{1+cI}$. The interpretation and values of parameters are described in the Table 1.

Remark 1. *In order to unify the existence of almost periodic solutions for SIR model with saturated and periodic incidence rate and saturated treatment function modelled by ordinary differential equations and their discrete analogues in the form of difference equations, combination of both continuous and discrete and to extend these results to more general time scales, we required much developed theory on time scales. Therefore, the qualitative study of (1) on time scales is challenging one.*

Let $\mathcal{C} = \mathcal{C}([-\tau, 0]_{\mathbb{T}}, \mathbb{R}^3)$ denote the Banach space and assume that the initial conditions of (1) satisfy

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta),$$

Table 1: Descriptions and values of parameters in model (1).

Parameter	Parameter description
A	The recruitment rate of the population
a, b, c	The auxiliary parameters
α	The natural mortality rate
d	The baseline contact rate
δ	The magnitude of forcing
γ	The natural recovery rate of the infective
β	The disease-related death rate

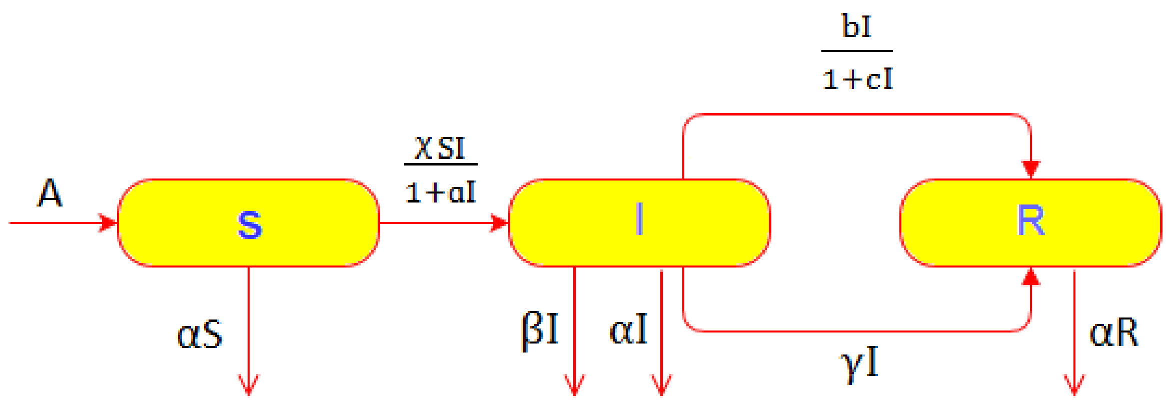

Figure 1: The transmission diagram.

$$\varphi_i(\theta) \geq 0, \; \theta \in [-\tau, 0], \; \varphi_i(0) > 0, i = 1, 2, 3,$$

where $(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{C}$. For a function $f(t)$ defined on $\mathbb{T}$, we denote

$$f^L = \inf\left\{f(t) : t \in \mathbb{T}\right\}, \; f^U = \sup\left\{f(t) : t \in \mathbb{T}\right\}.$$

Throughout the paper we suppose the following hold:

(H_1) $A, a, b, c, \alpha, \beta, \gamma : \mathbb{T} \to [0, \infty]$ are bounded almost periodic functions and satisfy $0 < A^L \leq A(t) \leq A^U$, $0 < a^L \leq a(t) \leq a^U$, $0 < b^L \leq b(t) \leq b^U$, $0 < c^L \leq c(t) \leq c^U$, $0 < \alpha^L \leq \alpha(t) \leq \alpha^U$, $0 < \beta^L \leq \beta(t) \leq \beta^U$, $0 < \gamma^L \leq \gamma(t) \leq \gamma^U$.

Next, we provide some definitions and lemmas which will be useful for later discussions.

Definition 1. *[2] A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by*

$$\sigma(t) = \inf\{\xi \in \mathbb{T} : \xi > t\}, \; \rho(t) = \sup\{\xi \in \mathbb{T} : \xi < t\}, \quad \textit{and} \;\; \mu(t) = \rho(t) - t,$$

respectively.

- *The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.*
- *A function $g : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.*

Definition 2. *[2] A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Also, we denote the set*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0, \forall t \in \mathbb{T}\}.$$

Lemma 1. *[8] If $a > 0, b > 0$ and $-b \in \mathcal{R}^+$. Then*

$$\mathtt{u}^{\Delta}(t) \leq (\geq) a - b\,\mathtt{u}(t),\ \ \mathtt{u}(t) > 0,\ \ t \in [t_0, \infty)_{\mathbb{T}},$$

implies

$$\mathtt{u}(t) \leq (\geq) \frac{a}{b}\Big[1 + \Big(\frac{b\,\mathtt{u}(t_0)}{a} - 1\Big) e_{(-b)}(t, t_0)\Big],\ \ t \in [t_0, \infty)_{\mathbb{T}}.$$

Definition 3. *[10] A time scale $\mathbb{T}$ is called an almost periodic time scale if*

$$\prod = \{\xi \in \mathbb{R} : t + \xi \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 4. *[10] Let $\mathbb{T}$ be an almost periodic time scale. Then a function $\mathtt{u} \in \mathcal{C}(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ε-translation set of w i.e.,*

$$\mathcal{E}\{\varepsilon, \mathtt{u}\} = \Big\{\xi \in \prod : |\mathtt{u}(t + \xi) - \mathtt{u}(t)| < \varepsilon, \forall t \in \mathbb{T}\Big\},$$

is a relatively dense set in $\mathbb{T}$ for any positive real number ε.

Definition 5. *[10] Let $\mathbb{D}$ be an open set of $\mathbb{R}^n$ and $\mathbb{T}$ be a positive almost periodic time scale. Then a function $\phi \in \mathcal{C}(\mathbb{T} \times \mathbb{D}, \mathbb{R}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $w \in \mathbb{D}$ if the ε-translation set of ϕ*

$$\mathcal{E}\{\varepsilon, \phi, \mathbb{S}\} = \Big\{\xi \in \prod : |\phi(t + \xi, \mathtt{u}) - \phi(t, \mathtt{u})| < \varepsilon, \forall (t, \mathtt{u}) \in \mathbb{T} \times \mathbb{S}\Big\},$$

is a relatively dense set in $\mathbb{T}$ for any positive real number ε, and for each compact subset $\mathbb{S}$ of $\mathbb{D}$. that is, for any given $\varepsilon > 0$ and each compact subset $\mathbb{S}$ of $\mathbb{D}$, there exists a constant $l(\varepsilon, \mathbb{S}) > 0$ such that each interval of length $l(\varepsilon, \mathbb{S})$ contains a $\xi(\varepsilon, \mathbb{S}) \in E\{\varepsilon, \phi, \mathbb{S}\}$ such that

$$|\phi(t + \xi, \mathtt{u}) - \phi(t, \mathtt{u})| < \varepsilon,\ \ \forall (t, \mathtt{u}) \in \mathbb{T} \times \mathbb{S}.$$

Next, consider the system

$$\varpi^{\Delta}(t) = \mathtt{g}(t, \varpi),\ t \in \mathbb{T}^+, \tag{2}$$

where $\mathtt{g} : \mathbb{T} \times \mathcal{S}_{\mathtt{M}} \to \mathbb{R}$, $\mathcal{S}_{\mathtt{M}} = \{\varpi \in \mathbb{R}^n : \|\varpi\| < \mathtt{M}\}$, $\|\varpi\| = \sup_{t\in\mathbb{T}} |\varpi(t)|$, $\mathtt{g}(t, \varpi)$ is almost periodic in t uniformly for $\varpi \in \mathcal{S}_{\mathtt{M}}$ and is continuous in ϖ. To find the solution of the (2), we consider the product system of (2) as follows:

$$\varpi^{\Delta}(t) = \mathtt{g}(t, \varpi),\ \ \ \vartheta^{\Delta}(t) = \mathtt{g}(t, \vartheta),$$

and we have the following lemma.

Lemma 2. *Let $\mathcal{V}(t,\varpi,\vartheta)$ be Lyapunov function defined on $\mathbb{T}^+\times\mathcal{S}_{\mathtt{M}}\times\mathcal{S}_{\mathtt{M}}$ and satisfies the following conditions*

(i) $\mathtt{A}\big(\|\varpi-\vartheta\|\big)\le\mathcal{V}(t,\varpi,\vartheta)\le\mathtt{B}\big(\|\varpi-\vartheta\|\big)$, *where* $\mathtt{A},\mathtt{B}\in\mathcal{P}$,

$$\mathcal{P}=\big\{\mathtt{G}\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+):\mathtt{G}(0)=0 \text{ and } \mathtt{G} \text{ is increasing}\big\};$$

(ii) $|\mathcal{V}(t,\varpi,\vartheta)-\mathcal{V}(t,\varpi_1,\vartheta_1)|\le\mathcal{L}\big(\|\varpi-\varpi_1\|+\|\vartheta-\vartheta_1\|\big)$, *where* $\mathcal{L}>0$ *is a constant;*

(iii) $\mathcal{D}^+\mathcal{V}^\Delta(t,\varpi,\vartheta)\le-\lambda\mathcal{V}(t,\varpi,\vartheta)$, *where* $\lambda>0,-\lambda\in\mathcal{R}^+$.

Further, if there exists a solution $\varpi(t)\in\mathcal{S}$ of system (2) for $t\in\mathbb{T}^+$, where $\mathbb{S}\subset\mathcal{S}_{\mathtt{M}}$ is a compact set, then there exist a unique almost periodic solution $\mathtt{p}(t)\in\mathcal{S}$ of system (2), which is uniformly asymptotically stable. Also, if $\mathtt{g}(t,\varpi)$ is periodic in t uniformly for $\varpi\in\mathcal{S}_{\mathtt{M}}$, then $\mathtt{p}(t)$ is also periodic.

Proof. Let $\{\ell_n\}$ be a sequence in $\prod$ such that $\ell_n\to+\infty$ as $n\to+\infty$. Suppose that $\psi\in\mathcal{S}$ is a solution of (2) for $t\in\mathtt{T}^+$, then $\psi(t+\ell_n)\in\mathcal{S}$ is a solution of the equation $\varpi^\Delta(t)=\mathtt{g}(t,\varpi)$. Let $\mathtt{U}$ be a compact subset of $\mathbb{T}$. Then, for any $\epsilon>0$, there exists large enough integer $\mathtt{N}(\epsilon)$ such that

$$e_{(-\lambda)}(\ell_k,0)<\frac{\mathtt{A}(\epsilon)}{2\mathtt{B}(2\mathtt{M})},\ \big\|\mathtt{g}(t+\ell_k,\varpi)-\mathtt{g}(t+\ell_m,\varpi)\big\|<\frac{\lambda\mathtt{A}(\epsilon)}{2\mathcal{L}},$$

whenever $m\ge k\ge\mathtt{N}(\epsilon)$. Then from (ii) and (iii), we have

$$\begin{aligned}\mathcal{D}^+\mathcal{V}^\Delta\big(t,\psi(t),\psi(t+\ell_m-\ell_k)\big)&\le-\lambda\mathcal{V}\big(t,\psi(t),\psi(t+\ell_m-\ell_k)\big)\\&\quad+\mathcal{L}\big\|\mathtt{g}\big(t+\ell_m-\ell_k,\psi(t+\ell_m-\ell_k)\big)-\mathtt{g}\big(t,\psi(t+\ell_m-\ell_k)\big)\big\|\\&\le-\lambda\mathcal{V}\big(t,\psi(t),\psi(t+\ell_m-\ell_k)\big)+\frac{\lambda\mathtt{A}(\epsilon)}{2}.\end{aligned}$$

Next for $m\ge k\ge\mathtt{N}(\epsilon)$, $t\in\mathtt{U}$ and from Lemma 1, we have

$$\begin{aligned}\mathcal{V}\big(t+\ell_k,\psi(t+\ell_k),\psi(t+\ell_m)\big)&\le e_{(-\lambda)}(t+\ell_k,0)\mathcal{V}\big(0,\psi(0),\psi(\ell_m-\ell_k))\big)\\&\quad+\frac{\mathtt{A}(\epsilon)}{2}\big(1-e_{(-\lambda)}(t+\ell_k,0)\big)\\&\le e_{(-\lambda)}(t+\ell_k,0)\mathcal{V}\big(0,\psi(0),\psi(\ell_m-\ell_k))\big)+\frac{\mathtt{A}(\epsilon)}{2}\\&<\frac{\mathtt{A}(\epsilon)}{2\mathtt{B}(2\mathtt{M})}\mathtt{B}(2\mathtt{M})+\frac{\mathtt{A}(\epsilon)}{2}=\mathtt{A}(\epsilon).\end{aligned}$$

By (i), for $m\ge k\ge\mathtt{N}(\epsilon)$ and $t\in\mathtt{U}$, we get $\|\psi(t+\ell_m)-\psi(t+\ell_k)\|<\epsilon$, which shows that $\psi(t)$ is asymptotically almost periodic. Then, $\psi(t)$ can be written as $\psi(t)=\mathtt{p}(t)+\mathtt{r}(t)$, where $\mathtt{p}(t)$ is almost periodic and $\mathtt{r}(t)\to0$ as $t\to0$. Thus, $\mathtt{p}(t)\in\mathcal{S}$ is an almost periodic solution of (2). Further, it can be proved easily that $\mathtt{p}(t)$ is uniformly asymptotically stable and every solution in $\mathcal{S}_{\mathtt{M}}$ tends to $\mathtt{p}(t)$, which means $\mathtt{p}(t)$ is unique. Moreover, if $\mathtt{g}(t,\varpi)$ is ω-periodic in t uniformly for $\varpi\in\mathcal{S}_{\mathtt{M}}$, $\mathtt{p}(t+\omega)\in\mathcal{S}$ is also a solution. By the uniqueness, we have $\mathtt{p}(t+\omega)=\mathtt{p}(t)$. This completes the proof. □

3 Permanence of solutions

In this section, we derive sufficient conditions for system (1) to be permanent.

Definition 6. *System (1) is said to be permanent if there are positive constants k, K such that*

$$k \le \liminf_{t\to\infty} S(t) \le \limsup_{t\to\infty} S(t) \le K,\ \ k \le \liminf_{t\to\infty} I(t) \ \le \limsup_{t\to\infty} I(t) \le K,$$
$$k \le \liminf_{t\to\infty} R(t) \le \limsup_{t\to\infty} R(t) \le K,$$

for any solution $(S(t), I(t), R(t))$ of system (1).

Lemma 3. *Assume that $(S(t), I(t), R(t))$ be a positive solution of system (1). If $-\alpha^L$, $-(\alpha^L + \beta^L + \gamma^L) \in \mathcal{R}^+$, there exist $T_3 > 0$ and $K > 0$ such that $S(t) \le K$, $I(t) \le K$, $R(t) \le K$ for $t \in [T_3, \infty)_{\mathbb{T}}$.*

Proof. Assume that $(S(t), I(t), R(t))$ be any positive solution of system (1). It follows from the first equation of system (1) that

$$S^{\Delta}(t) \le A(t) - \alpha(t)S(t) \le A^U - \alpha^L S(t).$$

Therefore, by Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$S(t) \le \frac{A^U}{\alpha^L} + \varepsilon := K_1, \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{3}$$

Next, from the second equation of system (1) and (3), for $t \in [T_1, \infty)$,

$$\begin{aligned} I^{\Delta}(t) &\le \frac{\chi(t)S(t)I(t-\tau)}{1+a(t)I(t-\tau)} - [\alpha(t)+\beta(t)+\gamma(t)]I(t) \\ &\le \frac{\chi(t)S(t)}{a(t)} - [\alpha(t)+\beta(t)+\gamma(t)]I(t) \le \frac{\chi^U K_1}{a^L} - [\alpha^L+\beta^L+\gamma^L]I(t). \end{aligned}$$

By Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_2 > T_1$ such that

$$I(t) \le \frac{\chi^U K_1}{a^L(\alpha^L+\beta^L+\gamma^L)} + \varepsilon := K_2, \quad t \in [T_2, \infty)_{\mathbb{T}}. \tag{4}$$

Finally, from the last equation of system (1) and (4), for $t \in [T_2, \infty]$,

$$\begin{aligned} R^{\Delta}(t) &= \gamma(t)I(t) + \frac{b(t)I(t)}{1+c(t)I(t)} - \alpha(t)R(t) \\ &\le \gamma(t)I(t) + \frac{b(t)}{c(t)} - \alpha(t)R(t) \le \left[\gamma^U K_2 + \frac{b^U}{c^L}\right] - \alpha^L R(t) \end{aligned}$$

By Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_3 > T_2$ such that

$$R(t) \le \frac{c^L\gamma^U K_2 + b^U}{c^L\alpha^L} + \varepsilon := K_3, \quad t \in [T_3, \infty).$$

Let $K > \max\{K_1, K_2, K_3\}$, then

$$S(t) \le K,\ I(t) \le K,\ R(t) \le K \quad for \quad t \in [T_3, \infty)_{\mathbb{T}}.$$

This completes the proof. ☐

Lemma 4. *Assume that* $(S(t), I(t), R(t))$ *be a positive solution of system* (1). *If*

$$A^L c^L > b^U, \ \frac{A^L c^L - b^U}{c^L(\alpha^U + \beta^U + \gamma^U)} > K,$$

and $-\alpha^U$, $-(\alpha^U + \beta^U + \gamma^U) \in \mathcal{R}^+$, *then there exist* $T_6 > 0$ *and* $k > 0$ *such that*

$$S(t) \geq k, \ I(t) \geq k, \ R(t) \geq k \quad for \quad t \in [T_6, \infty)_{\mathbb{T}}.$$

Proof. Assume that $(S(t), I(t), R(t))$ be any positive solution of system (1). It follows from the first equation of system (1) and Lemma 3 that, for $t \in [T_3, \infty)$,

$$S^\Delta(t) \geq A^L - \alpha^U S(t) - \frac{\chi^U K S(t)}{1 + a^L K} \geq A^L - \Big[\frac{\alpha^U + (\alpha^U a^L + \chi^U)K}{1 + a^L K}\Big] S(t).$$

Therefore, by Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_4 > 0$ such that

$$S(t) \geq \frac{A^L(1 + a^L K)}{\alpha^U + (\alpha^U a^L + \chi^U)K} + \varepsilon := k_1, \quad t \in [T_4, \infty)_{\mathbb{T}}. \tag{5}$$

Next, define $P(t) = S(t) + I(t), \ t \in [T_4, \infty)_{\mathbb{T}}$, and calculating the delta derivative of $P(t)$ along the solutions of (1), we have

$$\begin{aligned} P^\Delta(t) &= A(t) - \alpha(t)S(t) - [\alpha(t) + \beta(t) + \gamma(t)]I(t) - \frac{b(t)I(t)}{1 + c(t)I(t)} \\ &\geq A(t) - [\alpha(t) + \beta(t) + \gamma(t)]S(t) - [\alpha(t) + \beta(t) + \gamma(t)]I(t) - \frac{b(t)I(t)}{1 + c(t)I(t)} \\ &\geq A(t) - [\alpha(t) + \beta(t) + \gamma(t)](I(t) + S(t)) - \frac{b(t)}{c(t)} \\ &\geq \Big[\frac{A^L c^L - b^U}{c^L}\Big] - [\alpha^U + \beta^U + \gamma^U]P(t). \end{aligned} \tag{6}$$

By Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_5 > T_4$, it follows from (6) that, for $t \in [T_5, \infty)$,

$$P(t) \geq \frac{A^L c^L - b^U}{c^L(\alpha^U + \beta^U + \gamma^U)} + \varepsilon.$$

From the definition of $P(t)$ and Lemma 3, it follows that

$$I(t) \geq \frac{A^L c^L - b^U}{c^L(\alpha^U + \beta^U + \gamma^U)} - K + \varepsilon := k_2. \tag{7}$$

By the third equation of the system (1), Lemma 3 and (7), for $t \in [T_5, \infty)$, we have

$$R^\Delta(t) \geq \Big[\gamma^L k_2 + \frac{b^L k_2}{1 + c^U K}\Big] - \alpha^U R(t). \tag{8}$$

By Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_6 > T_5$, it follows from (8) that, for $t \in [T_6, \infty)$,

$$R(t) \geq \frac{[\gamma^L(1 + c^U K) + b^L]k_2}{\alpha^U(1 + c^U K)} + \varepsilon := k_3.$$

Let $0 < k < \min\{k_1, k_2, k_3\}$, then $S(t) \geq k, \ I(t) \geq k, \ R(t) \geq k \ for \ t \in [T_6, \infty)_{\mathbb{T}}$. □

Theorem 1. *Assume that the conditions of Lemma 3 and Lemma 4 hold, then system (1) is permanent.*

Proof. Together with Lemma 3 and 4, we can obtain desired result. □

Define

$$\Omega = \Big\{ \big(S(t), I(t), R(t)\big) : \big(S(t), I(t), R(t)\big) \text{be a solution of (1) and}$$
$$0 < s_* \leq S(t) \leq s^*,\ 0 < i_* \leq I(t) \leq i^*,\ 0 < r_* \leq R(t) \leq r^* \Big\}.$$

It is clear that Ω is invariant set of system (1).

Lemma 5. *If hypothesis of Lemmas 3 and 4 holds. Then $\Omega \neq \emptyset$.*

Proof. It can be easily proved. So, we omit it here. □

4 Uniform asymptotic stability

In this section, we establish sufficient conditions for the existence and uniform asymptotic stability of the unique positive almost periodic solution to system (1).

Theorem 2. *If (H_1) and the following holds:*

(H_2) $\eta > 0$ *and* $-\eta \in \mathcal{R}^+$, *where* $\eta = \min\{\eta_1, \eta_2, \alpha^L\}$ *where*

$$\eta_1 = \alpha^L + \frac{a^L\chi^L i_*^2 + \chi^L i_*}{(1 + a^U i^*)^2} - \frac{a^U\chi^U i^{*2} + \chi^U i^*}{(1 + a^L i_*)^2},$$
$$\eta_2 = (\alpha^L + \beta^L + \gamma^L) + \frac{b^L}{(1 + c^U i^*)^2} + \frac{\chi^L s_*}{(1 + a^U i^*)^2} - \frac{b^U}{(1 + c^L i_*)^2} - \frac{\chi^U s^*}{(1 + a^L i_*)^2} - \gamma^U$$

then the dynamic system (1) has a unique almost periodic solution $\big(S(t), I(t), R(t)\big) \in \Omega$ and is uniformly asymptotically stable.

Proof. According to Theorem 1 and Lemma 5, every solution $\big(S(t), I(t), R(t)\big)$ of system (1) satisfies $s_* \leq S(t) \leq s^*$, $i_* \leq I(t) \leq i^*$, $r_* \leq R(t) \leq r^*$. Hence, $|S(t)| \leq A_i$, $|I(t)| \leq B_i$, $|R(t)| \leq C_i$ where $A_i = \max\{|s_*|, |s^*|\}$, $B_i = \max\{|i_*|, |i^*|\}$ and and $C_i = \max\{|r_*|, |r^*|\}$.

$$\text{Denote } \|\big(S(t), I(t), R(t)\big)\| = \sup_{t\in\mathbb{T}^+} |S(t)| + \sup_{t\in\mathbb{T}^+} |I(t)| + \sup_{t\in\mathbb{T}^+} |R(t)|.$$

Suppose that $X = \big(S(t), I(t), R(t)\big)$, $\hat{X} = \big(\hat{S}(t), \hat{I}(t), \hat{R}(t)\big)$ are any two positive solutions of system (1), then

$$\|X\| \leq A + B + C \quad \text{and} \quad \|\hat{X}\| \leq A + B + C.$$

In view of system (1), we have

$$\left.\begin{aligned}
\hat{S}^\Delta(t) &= A(t) - \alpha(t)\hat{S}(t) - \frac{\chi(t)\hat{S}(t)\hat{I}(t-\tau)}{1 + a(t)\hat{I}(t-\tau)},\\
\hat{I}^\Delta(t) &= \frac{\chi(t)\hat{S}(t)\hat{I}(t-\tau)}{1 + a(t)\hat{I}(t-\tau)} - [\alpha(t) + \beta(t) + \gamma(t)]\hat{I}(t) - \frac{b(t)\hat{I}(t)}{1 + c(t)\hat{I}(t)},\\
\hat{R}^\Delta(t) &= \gamma(t)\hat{I}(t) + \frac{b(t)\hat{I}(t)}{1 + c(t)\hat{I}(t)} - \alpha(t)\hat{R}(t).
\end{aligned}\right\} \tag{9}$$

Define the Lyapunov function $\mathcal{V}(t, X, \hat{X})$ on $\mathbb{T}^+ \times \Omega \times \Omega$ as

$$\mathcal{V}(t, X, \hat{X}) = |S(t) - \hat{S}(t)| + |I(t) - \hat{I}(t)| + |R(t) - \hat{R}(t)|.$$

Define the norm

$$\|X(t) - \hat{X}(t)\| = \sup_{t \in \mathbb{T}^+} |S(t) - \hat{S}(t)| + \sup_{t \in \mathbb{T}^+} |I(t) - \hat{I}(t)| + \sup_{t \in \mathbb{T}^+} |R(t) - \hat{R}(t)|.$$

It is easy to see that there exist two constants $l > 0$, $m > 0$ such that

$$l\|X(t) - \hat{X}(t)\| \leq V(t, X, \hat{X}) \leq m\|X(t) - \hat{X}(t)\|.$$

Let $\mathtt{A}, \mathtt{B} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$, $\mathtt{A}(x) = lx$, $\mathtt{B}(x) = mx$, then the assumption (i) of Lemma 2 is satisfied. On the other hand, we have

$$\begin{aligned}
\big|\mathcal{V}\big(t, X(t), \hat{X}(t)\big) - \mathcal{V}\big(t, X^*(t), \hat{X}^*(t)\big)\big| &= \big||S(t) - \hat{S}(t)| + |I(t) - \hat{I}(t)| + |R(t) - \hat{R}(t)| \\
&\quad -|S^*(t) - \hat{S}^*(t)| - |I^*(t) - \hat{I}^*(t)| - |R^*(t) - \hat{R}^*(t)|\big| \\
&\leq \big|S(t) - S^*(t)\big| + \big|I(t) - I^*(t)\big| + \big|R(t) - R^*(t)\big| \\
&\quad +\big|\hat{S}(t) - \hat{S}^*(t)\big| + \big|\hat{I}(t) - \hat{I}^*(t)\big| + +\big|\hat{R}(t) - \hat{R}^*(t)\big| \\
&= L\big[\|X - X^*(t)\| + \|\hat{X}(t) - \hat{X}^*(t)\|\big],
\end{aligned}$$

where $\mathcal{L} = 1$, so condition (ii) of Lemma 2 is satisfied. Now consider a function $\mathcal{W}(t) = \mathcal{W}_1(t) + \mathcal{W}_2(t) + \mathcal{W}_3(t)$, where

$$\mathcal{W}_1(t) = |S(t) - \hat{S}(t)|, \ \mathcal{W}_2(t) = |I(t) - \hat{I}(t)|,$$

and

$$\mathcal{W}_3(t) = |R(t) - \hat{R}(t)| + \left[\frac{\chi^U s^*}{(1 + a^L i_*)^2} - \frac{\chi^L s_*}{(1 + a^U i^*)^2}\right] \int_{t-\tau}^{t} |I(t) - \hat{I}(t)| \Delta t.$$

For $t \in \mathbb{T}^+$, calculating the delta derivative $D^+\mathcal{W}_1(t)^\Delta$ of $W_1(t)$ along system (9), we get

$$\begin{aligned}
D^+\mathcal{W}_1^\Delta(t) &\leq sign\big(S(\sigma(t)) - \hat{S}(\sigma(t))\big)\big[S(t) - \hat{S}(t)\big]^\Delta \\
&\leq sign\big(S(\sigma(t)) - \hat{S}(\sigma(t))\big)\bigg[-\alpha(t)\big(S(t) - \hat{S}(t)\big) - \frac{\chi(t)S(t)I(t-\tau)}{1 + a(t)I(t-\tau)} \\
&\qquad + \frac{\chi(t)\hat{S}(t)\hat{I}(t-\tau)}{1 + a(t)\hat{I}(t-\tau)}\bigg] \\
&\leq sign\big(S(\sigma(t)) - \hat{S}(\sigma(t))\big)\bigg[-\alpha(t)\big(S(t) - \hat{S}(t)\big) \\
&\qquad - \frac{(a(t)\chi(t)I(t-\tau)\hat{I}(t-\tau) + \chi(t)\hat{I}(t-\tau))}{(1 + a(t)I(t-\tau))(1 + a(t)\hat{I}(t-\tau))}\big(S(t) - \hat{S}(t)\big) \\
&\qquad - \frac{\chi(t)S(t)}{(1 + a(t)I(t-\tau))(1 + a(t)\hat{I}(t-\tau))}\big(I(t-\tau) - \hat{I}(t-\tau)\big)\bigg] \\
&\leq -\left[\alpha^L + \frac{a^L\chi^L i_*^2 + \chi^L i_*}{(1 + a^U i^*)^2}\right]\big|S(t) - \hat{S}(t)\big| - \frac{\chi^L s_*}{(1 + a^U i^*)^2}\big|I(t-\tau) - \hat{I}(t-\tau)\big|.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
D^{+}\mathcal{W}_2^{\Delta}(t) &\le sign\big(I(\sigma(t)) - \hat{I}(\sigma(t))\big)\big[I(t) - \hat{I}(t)\big]^{\Delta} \\
&\le sign\big(I(\sigma(t)) - \hat{I}(\sigma(t))\big)\bigg[\frac{\chi(t)S(t)I(t-\tau)}{1+a(t)I(t-\tau)} - \frac{\chi(t)\hat{S}(t)\hat{I}(t-\tau)}{1+a(t)\hat{I}(t-\tau)} \\
&\quad - \big(\alpha(t)+\beta(t)+\gamma(t)\big)\big(I(t)-\hat{I}(t)\big) - \frac{b(t)I(t)}{1+c(t)I(t)} + \frac{b(t)\hat{I}(t)}{1+c(t)\hat{I}(t)}\bigg] \\
&\le sign\big(I(\sigma(t)) - \hat{I}(\sigma(t))\big)\bigg[\frac{(a(t)\chi(t)I(t-\tau)\hat{I}(t-\tau) + \chi(t)\hat{I}(t-\tau))}{(1+a(t)I(t-\tau))(1+a(t)\hat{I}(t-\tau))}\big(S(t)-\hat{S}(t)\big) \\
&\quad + \frac{\chi(t)S(t)\big(I(t-\tau)-\hat{I}(t-\tau)\big)}{(1+a(t)I(t-\tau))(1+a(t)\hat{I}(t-\tau))} - \big(\alpha(t)+\beta(t)+\gamma(t)\big)\big(I(t)-\hat{I}(t)\big) \\
&\quad - \frac{b(t)}{(1+c(t)I(t))(1+c(t)\hat{I}(t))}\big(I(t)-\hat{I}(t)\big)\bigg] \\
&\le \bigg[\frac{a^U\chi^U i^{*2} + \chi^U i^*}{(1+a^L i_*)^2}\bigg]\big|S(t)-\hat{S}(t)\big| + \frac{\chi^U s^*}{(1+a^L i_*)^2}\big|I(t-\tau)-\hat{I}(t-\tau)\big| \\
&\quad - \bigg[(\alpha^L+\beta^L+\gamma^L) + \frac{b^L}{(1+c^U i^*)^2}\bigg]\big|I(t)-\hat{I}(t)\big|,
\end{aligned}
$$

and

$$
\begin{aligned}
D^{+}\mathcal{W}_3^{\Delta}(t) &\le sign\big(R(\sigma(t)) - \hat{R}(\sigma(t))\big)\big[R(t)-\hat{R}(t)\big]^{\Delta} \\
&\quad + \bigg[\frac{\chi^U s^*}{(1+a^L i_*)^2} - \frac{\chi^L s_*}{(1+a^U i^*)^2}\bigg]\Big[|I(t)-\hat{I}(t)| - |I(t-\tau)-\hat{I}(t-\tau)|\Big] \\
&\le sign\big(R(\sigma(t)) - \hat{R}(\sigma(t))\big)\bigg[\gamma(t)\big(I(t)-\hat{I}(t)\big) \\
&\quad + \frac{b(t)I(t)}{1+c(t)I(t)} - \frac{b(t)\hat{I}(t)}{1+c(t)\hat{I}(t)} - \alpha(t)\big(R(t)-\hat{R}(t)\big)\bigg] \\
&\le \bigg[\gamma^U + \frac{b^U}{(1+c^L i_*)^2} + \frac{\chi^U s^*}{(1+a^L i_*)^2} - \frac{\chi^L s_*}{(1+a^U i^*)^2}\bigg]\big|I(t)-\hat{I}(t)\big| \\
&\quad - \bigg[\frac{\chi^U s^*}{(1+a^L i_*)^2} - \frac{\chi^L s_*}{(1+a^U i^*)^2}\bigg]\big|I(t-\tau)-\hat{I}(t-\tau)\big| - \alpha^L\big|R(t)-\hat{R}(t)\big|.
\end{aligned}
$$

Since $\mathcal{V}(t) \le \mathcal{W}(t)$ for $t \in \mathbb{T}^{+}$ and by assumption (H_2), it follows that

$$
\begin{aligned}
D^{+}\big(\mathcal{V}(t)\big)^{\Delta} &\le D^{+}\big(\mathcal{W}(t)\big)^{\Delta} = D^{+}\big(\mathcal{V}_1(t)+\mathcal{V}_2(t)+\mathcal{V}_3(t)\big)^{\Delta} \\
&\le -\bigg[\alpha^L + \frac{a^L\chi^L i_*^2 + \chi^L i_*}{(1+a^U i^*)^2} - \frac{a^U\chi^U i^{*2} + \chi^U i^*}{(1+a^L i_*)^2}\bigg]\big|S(t)-\hat{S}(t)\big| \\
&\quad - \bigg[(\alpha^L+\beta^L+\gamma^L) + \frac{b^L}{(1+c^U i^*)^2} + \frac{\chi^L s_*}{(1+a^U i^*)^2} \\
&\quad - \frac{b^U}{(1+c^L i_*)^2} - \frac{\chi^U s^*}{(1+a^L i_*)^2} - \gamma^U\bigg]\big|I(t)-\hat{I}(t)\big| - \alpha^L\big|R(t)-\hat{R}(t)\big|
\end{aligned}
$$

$$\leq -\eta_1\big|S(t)-\hat{S}(t)\big| - \eta_2\big|I(t)-\hat{I}(t)\big| - \alpha^L\big|R(t)-\hat{R}(t)\big|$$
$$\leq -\eta V(t).$$

By (H_2), we see that Condition (iii) of Lemma 2 is satisfied. Hence, according to Lemma 2, there exists a unique uniformly asymptotically stable almost periodic solution $\big(S(t), I(t), R(t)\big)$ of system (1) and $\big(S(t), I(t), R(t)\big) \in \Omega$. The proof is complete. □

5 Numerical Simulations

In this section we provide some numerical simulations to illustrate the results obtained in the previous sections.

Example 1. Consider the dynamic susceptible-infected-recovered (SIR) model with saturated treatment on time scale $\mathbb{T}^+$:

$$\left.\begin{aligned} S^{\Delta}(t) &= A(t) - \alpha(t)S(t) - \frac{\chi(t)S(t)I(t-0.004)}{1+a(t)I(t-0.004)}, \\ I^{\Delta}(t) &= \frac{\chi(t)S(t)I(t-0.004)}{1+a(t)I(t-0.004)} - [\alpha(t)+\beta(t)+\gamma(t)]I(t) - \frac{b(t)I(t)}{1+c(t)I(t)}, \\ R^{\Delta}(t) &= \gamma(t)I(t) + \frac{b(t)I(t)}{1+c(t)I(t)} - \alpha(t)R(t), \end{aligned}\right\} \tag{10}$$

where $A(t) = 0.5 + |\sin\sqrt{2}t|$, $\alpha = 5 + |\cos\sqrt{5}t|$, $\beta = 0.1$, $\gamma = 0.02 + |\sin\pi t|$, $a(t) = 0.5$, $b(t) = 0.1$, $c(t) = 0.05$, $\chi(t) = 2\times10^{-3} + 2\times10^{-4}\sin((\pi/6)t)$. By direct calculations, we obtain $s^* = 0.3$, $i^* = 0.0004327868853$, $r^* = 0.1715168599$, and $s_* = 0.3446911567$, $i_* = 0.6900990099$, $r_* = 0.01363200507$. Let $K = 0.4$. Then $K > \max\{s^*, i^*, r^*\} = 0.3$, $A^Lc^L - b^U = 0.25 > 0$ and

$$\frac{A^Lc^L - b^U}{c^L(\alpha^U + \beta^U + \gamma^U)} = 0.9900990099 > K.$$

Therefore, by Theorem 1, (10) is permanent.

Now by these values, we get $\eta_1 = 5.001855105, \eta_2 = 4.106533372$. S0, $\eta = \min\{\eta_1, \eta_2, \alpha^L\} = \eta_2 > 0$. By Theorem 2, (10) has a unique almost periodic solution $\big(S(t), I(t), R(t)\big) \in \Omega$ and is uniformly asymptotically stable. From Fig. 2-5, we can see that for system (10), there exists a positive almost periodic solution denoted by $\big(S^*(t), I^*(t), R^*(t)\big)$. Moreover, Fig. 6-8 shows that any positive solution $\big(S(t), I(t), R(t)\big)$ tends to the above almost periodic solution $\big(S^*(t), I^*(t), R^*(t)\big)$.

In addition, from Fig. 1-8 when the initial conditions are different, the disease will tend toward different periodic solutions. So, besides related control measures, we can change the initial condition to change the tendency of the disease.

6 Conclusion

In the real nature, due to the interference of various factors, such as seasonal effects of the weather, food supplies, and mating habits, the coefficients of most of the systems are approximate to certain periodic functions. However, with the uncertainty of the interferences, the

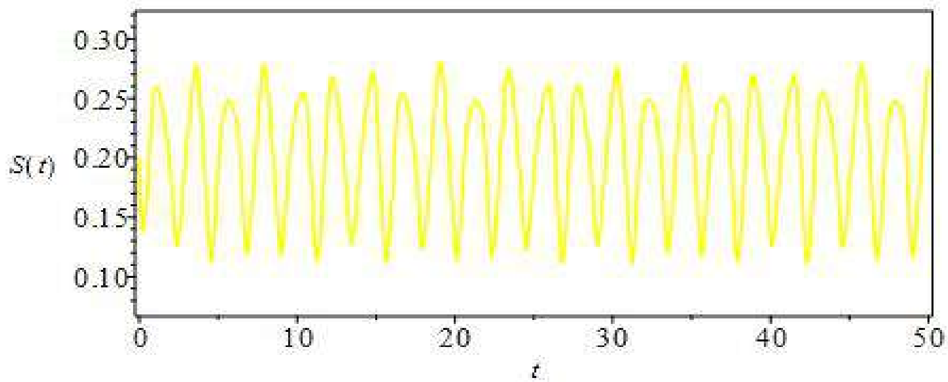

Figure 2: Positive almost periodic solution of system (10). Time series of $S(t)$ with initial value $S(0) = 0.12$ and t over $[0, 50]$.

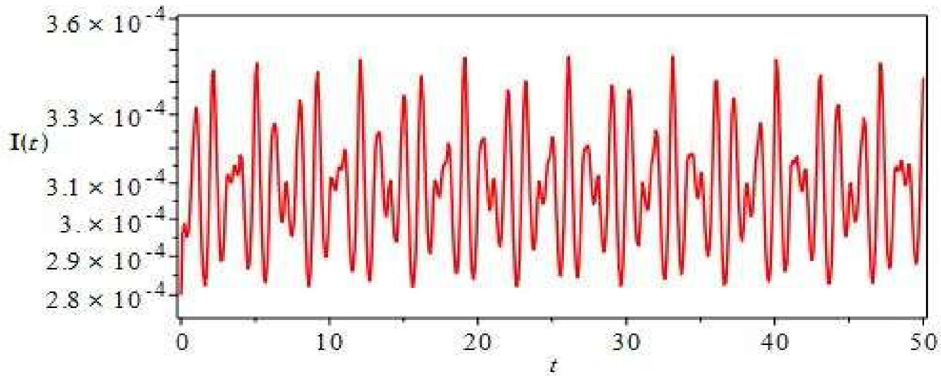

Figure 3: Positive almost periodic solution of system (10). Time series of $I(t)$ with initial value $I(0) = 0.00028$ and t over $[0, 50]$.

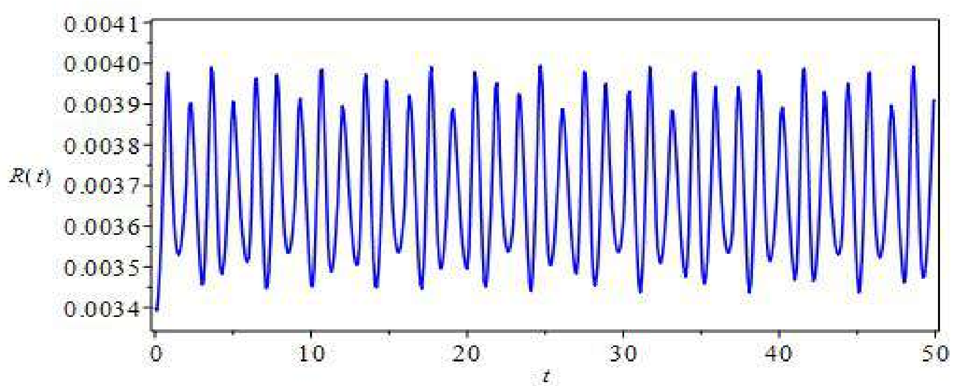

Figure 4: Positive almost periodic solution of system (10). Time series of $R(t)$ with initial value $R(0) = 0.0034$ and t over $[0, 50]$.

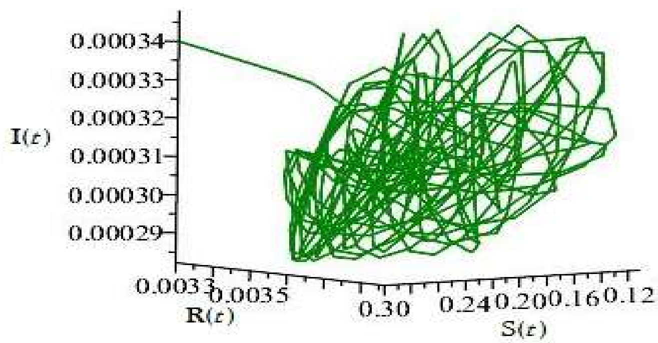

Figure 5: Positive almost periodic solution of system (10). Time series of $\big(S(t), I(t), R(t)\big)$ with initial value $\big(S(0), I(0), R(0)\big) = (0.3, 0.00034, 0.0033)$.

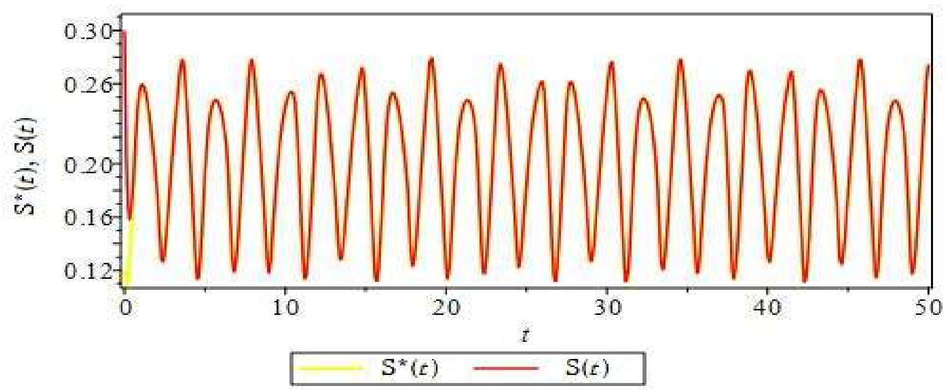

Figure 6: Uniformly asymptotic stability of system (10). Time series of $S^*(t)$ and $S(t)$ with initial values $S^*(0) = 0.12$, $S(0) = 0.3$, and t over $[0, 50]$.

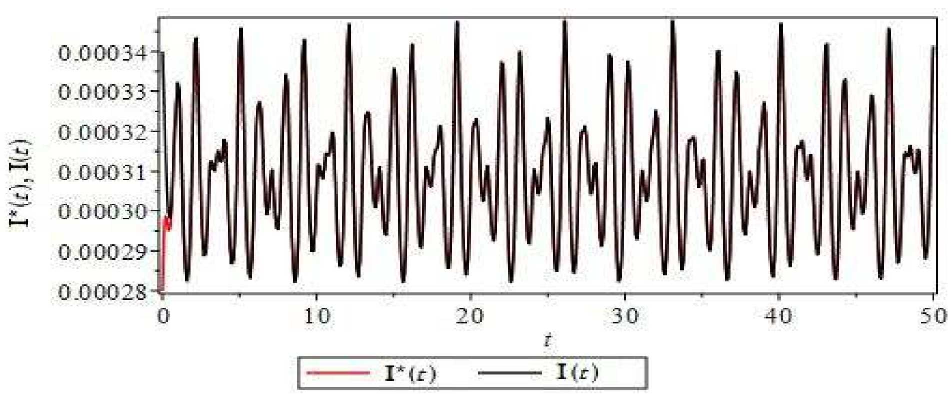

Figure 7: Uniformly asymptotic stability of system (10). Time series of $I^*(t)$ and $I(t)$ with initial values $I^*(0) = 0.00034$, $I(0) = 0.00028$, and t over $[0, 50]$.

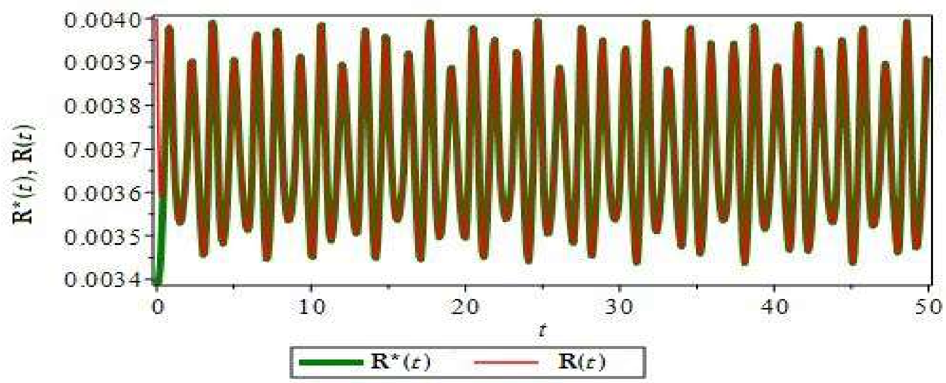

Figure 8: Uniformly asymptotic stability of system (10). Time series of $R^*(t)$ and $R(t)$ with initial values $R^*(0) = 0.004$, $R(0) = 0.0034$, and t over $[0, 50]$.

coefficients of the systems are not strictly periodic. Therefore, almost periodicity is a more common phenomenon than strict periodicity. Hence, we dealt with the almost periodic dynamics of a time-delayed SIR epidemic model with saturated treatment on time scales. By establishing some dynamic inequalities on time scales, a permanence result for the model is obtained. Furthermore, by means of the almost periodic functional theory on time scales and Lyapunov functional, some criteria is obtained for the existence, uniqueness and uniform asymptotic stability of almost periodic solutions of the model. Thus, the mathematical results in the paper are quite new, and it may have some application value and practical significance for the prediction and control strategy for corresponding ecoepidemic systems. Our future research will focus on the stability of the periodic solution and apply our mathematical methods to the research of special diseases.

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Denumerably Many Positive Solutions for Iterative Systems of Singular Two-Point Boundary Value Problems on Time Scales

K. R. Prasad and **Mahammad Khuddush**
Department of Applied Mathematics
College of Science and Technology, Andhra University
Visakhapatnam, 530003, India
rajendra92@rediffmail.com
khuddush89@gmail.com

K. V. Vidyasagar[3]
Department of Mathematics
Government Degree College for Women
Marripalem, Koyyuru Mandal, Visakhapatnam, 531116, India
Vidyavijaya08@gmail.com

Abstract

In this paper we consider a dynamical iterative system of two-point boundary value problems with integral boundary conditions, having n singularities and involving an increasing homeomorphism, positive homomorphism operator. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of denumerably many positive solutions. Finally we provide an example to check validity of our obtained results.

AMS Subject Classifications: 34N05, 34B16, 34B18.
Keywords: Iterative system, singularity, homeomorphism, homomorphism, cone, positive solutions, time scale, Krasnoselskii's fixed point theorem.

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1 Introduction

The differential, difference and dynamic equations on time scales are three equations play important role for modelling in the environment. Among them, the theory of dynamic equations on time scales is the most recent and was introduced by Stefan Hilger in his PhD thesis in 1988 [13] with three main features: unification, extension and discretization. Since a time scale is any closed and nonempty subset of the real numbers set. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles Agarwal and Bohner [1], Agarwal et al. [2] and monographs of Bohner and Peterson [6, 7].

The study of turbulent flow through porous media is important for a wide range of scientific and engineering applications such as fluidized bed combustion, compact heat exchangers, combustion in an inert porous matrix, high temperature gas-cooled reactors, chemical catalytic reactors [8] and drying of different products such as iron ore [16]. To study such type of problems, Leibenson [14] introduced the following p-Laplacian equation,

$$\big(\phi_p(\vartheta'(t))\big)' = f\big(t, \vartheta(t), \vartheta'(t)\big),$$

where $\phi_p(\vartheta) = |\vartheta|^{p-2}\vartheta,\ p > 1$, is the p-Laplacian operator its inverse function is denoted by $\phi_q(\tau)$ with $\phi_q(\tau) = |\tau|^{q-2}\tau$, and p, q satisfy $1/p+1/q = 1$. It is well known fact that the p-Laplacian operator and fractional calculus arises from many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science, it is worth studying the fractional differential equations with p-Laplacian operator.

In this paper, we consider an operator ϕ called increasing homeomorphism and positive homomorphism operator(IHPHO), which generalizes and improves the p-Laplacian operator for some $p > 1$, and ϕ is not necessarily odd. Liang and Zhang [15] studied countably many positive solutions for nonlinear singular m–point boundary value problems on time scales with IHPHO,

$$\begin{aligned}
&\big(\phi(\vartheta^{\Delta}(t))\big)^{\nabla} + a(t)f\big(\vartheta(t)\big) = 0,\ t \in [0, T]_{\mathbb{T}}\\
&\vartheta(0) = \sum_{i=1}^{m-2} a_i\vartheta(\xi_i),\ \vartheta^{\Delta}(T) = 0,
\end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [9], Dogan considered second order p–Laplacian boundary value problem on time scales,

$$\begin{aligned} &\big(\phi_p(\vartheta^{\Delta}(t))\big)^{\nabla} + \omega(t) f\big(t, \vartheta(t)\big) = 0,\ t \in [0, T]_{\mathbb{T}} \\ &\vartheta(0) = \sum_{i=1}^{m-2} a_i \vartheta(\xi_i),\ \phi_p(\vartheta^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi_p(\vartheta^{\Delta}(\xi_i)), \end{aligned}$$

and established existence of multiple positive solutions by applying fixed-point index theory.

Inspired by aforementioned works, in this paper by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of denumerably many positive solutions for dynamical iterative system of two-point boundary value problem with n singularities and involving IHPHO on time scales,

$$\left.\begin{aligned} \phi\big(\vartheta_j^{\Delta\nabla}(t)\big) + \zeta(t) f_j\big(\vartheta_{j+1}(t)\big) = 0,\ 1 \le j \le \ell,\ t \in [0, \mathfrak{T}]_{\mathbb{T}} \\ \vartheta_{\ell+1}(t) = \vartheta_1(t),\ t \in [0, \mathfrak{T}]_{\mathbb{T}}, \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \vartheta_j(0) = \int_0^{\mathfrak{T}} \kappa(s) \vartheta_j(s) \nabla s,\ 1 \le j \le \ell, \\ \vartheta_j(\mathfrak{T}) = \int_0^{\mathfrak{T}} \kappa(s) \vartheta_j(s) \nabla s,\ 1 \le j \le \ell, \end{aligned}\right\} \tag{1.2}$$

where $\ell \in \mathbb{N}$, $\zeta(t) = \prod_{i=1}^{n} \zeta_i(t)$ and each $\zeta_i(t) \in L_{\nabla}^{p_i}([0, \mathfrak{T}]_{\mathbb{T}})(p_i \ge 1)$ has a singularity in the interval $\left(0, \dfrac{\mathfrak{T}}{2}\right)$ and $\phi : \mathbb{R} \to \mathbb{R}$ is an IHPHO with $\phi(0) = 0$.

A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called a IHPHO, if the following three conditions are fulfilled,

(a) $\phi(\tau_1) \le \phi(\tau_2)$ whenever $\tau_1 \le \tau_2$, for any real numbers τ_1, τ_2.

(b) ϕ is a continuous bijection and its inverse ϕ^{-1} is continuous.

(c) $\phi(\tau_1\tau_2) = \phi(\tau_1)\phi(\tau_2)$ for any real numbers τ_1, τ_2.

We assume the following conditions are true in the entire paper:

(H_1) $f_j : [0, +\infty) \to [0, +\infty)$ and $\kappa : [0, \mathfrak{T}]_{\mathbb{T}} \to [0, +\infty)$ are continuous,

(H_2) there exists a sequence $\{t_r\}_{r=1}^{\infty}$ such that $0 < t_{r+1} < t_r < \dfrac{\mathfrak{T}}{2}$,

$$\lim_{r\to\infty} t_r = t^* < \frac{\mathfrak{T}}{2},\ \lim_{t\to t_r} \zeta_i(t) = +\infty,\ i, r \in \mathbb{N}$$

and each $\zeta_i(t)$ does not vanish identically on any subinterval of $[0, \mathfrak{T}]_{\mathbb{T}}$. Moreover, there exists $\delta_i > 0$ such that

$$\delta_i < \phi^{-1}\left(\zeta_i(t)\right) < \infty\ a.e.\ \text{on}\ [0, \mathfrak{T}]_{\mathbb{T}},\ i = 1, 2, \cdots, n.$$

2 Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions; for details, see [3–6, 11, 18, 19].

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$, and $\mu(t) = \rho(t) - t$, respectively.

- The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.
- If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.
- By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ other intervals can be defined similarly.

Definition 2.2. Let μ_Δ and μ_∇ be the Lebesgue $\Delta-$ measure and the Lebesgue $\nabla-$ measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.

(i) If there exists $E_1 \subset A$ with $\mu_\Delta(E_1) = 0$ such that P holds on $A\backslash E_1$, then P is said to hold Δ–a.e. on A.

(ii) If there exists $E_2 \subset A$ with $\mu_\nabla(E_2) = 0$ such that P holds on $A\backslash E_2$, then P is said to hold ∇–a.e. on A.

Definition 2.3. Let $E \subset \mathbb{T}$ be a ∇–measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be ∇–measurable function. We say that f belongs to $L^p_\nabla(E)$ provided that either

$$\int_E |f|^p(s)\nabla s < \infty \quad \text{if} \quad p \in \mathbb{R},$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M, \ \ \nabla - a.e. \ on \ E \ \ \text{if} \ \ p = +\infty.$$

Lemma 2.4. *Let $E \subset \mathbb{T}$ be a ∇–measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is a ∇–integrable on E, then*

$$\int_E f(s)\nabla s = \int_E f(s)ds + \sum_{i\in I_E} \big(t_i - \rho(t_i)\big) f(t_i),$$

where $I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i\in I}, I \subset \mathbb{N}$, is the set of all left-scattered points of $\mathbb{T}$.

Lemma 2.5. *Suppose $0 < \eta < 1$, where $\eta = \displaystyle\int_0^{\mathfrak{T}} \kappa(\tau)\nabla\tau$. Then for any $\varrho(t) \in C([0,\mathfrak{T}]_{\mathbb{T}})$, boundary value problem,*

$$-\phi(\vartheta_1^{\Delta\nabla}(t)) = \varrho(t),\ t \in [0,\mathfrak{T}]_{\mathbb{T}}, \tag{2.1}$$

$$\vartheta_1(0) = \vartheta_1(\mathfrak{T}) = \int_0^{\mathfrak{T}} \kappa(\tau)\vartheta_1(\tau)\nabla\tau, \tag{2.2}$$

has a unique solution

$$\vartheta_1(t) = \int_0^{\mathfrak{T}} \aleph(t,\tau)\phi^{-1}(\varrho(\tau))\nabla\tau, \tag{2.3}$$

where

$$\aleph(t,\tau) = \aleph_0(t,\tau) + \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1, \tag{2.4}$$

in which

$$\aleph_0(t,\tau) = \frac{1}{\mathfrak{T}}\begin{cases} t(\mathfrak{T}-\tau), & t \le \tau, \\ \tau(\mathfrak{T}-t), & \tau \le t. \end{cases} \tag{2.5}$$

Proof. Suppose ϑ_1 is a solution of (2.1), then

$$\begin{aligned}\vartheta_1(t) &= -\int_0^t\int_0^{\tau} \phi^{-1}(\varrho(\tau_1))\nabla\tau_1\Delta\tau + At + B \\ &= -\int_0^t (t-\tau)\phi^{-1}(\varrho(\tau))\nabla\tau + A_1 t + A_2,\end{aligned}$$

where $A_1 = \vartheta_1^{\Delta}(0)$ and $A_2 = \vartheta_1(0)$. By the conditions (2.2), we get

$$A_1 = \frac{1}{\mathfrak{T}}\int_0^{\mathfrak{T}} (\mathfrak{T}-\tau)\phi^{-1}(\varrho(\tau))\nabla\tau,$$

and

$$\begin{aligned}
A_2 &= \int_0^{\mathfrak{T}} \kappa(\tau)\vartheta_1(\tau)\nabla\tau \\
&= \int_0^{\mathfrak{T}} \kappa(\tau)\Big[-\int_0^{\tau}(\tau-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1 + A_1\tau + A_2\Big]\nabla\tau \\
&= \int_0^{\mathfrak{T}} \kappa(\tau)\Big[-\int_0^{\tau}(\tau-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1 \\
&\qquad\qquad + \frac{\tau}{\mathfrak{T}}\int_0^{\mathfrak{T}}(\mathfrak{T}-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1\Big]\nabla\tau + A_2\eta \\
&= \int_0^{\mathfrak{T}} \kappa(\tau)\Big[\int_0^{\tau}\frac{s}{\mathfrak{T}}(\mathfrak{T}-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1 \\
&\qquad\qquad + \int_{\tau}^{\mathfrak{T}}\frac{\tau}{\mathfrak{T}}(\mathfrak{T}-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1\Big]\nabla\tau + A_2\eta \\
&= \int_0^{\mathfrak{T}} \kappa(\tau)\Big[\int_0^{\mathfrak{T}}\aleph_0(\tau,s)\phi^{-1}(\varrho(\tau))\nabla\tau\Big]\nabla\tau + A_2\eta \\
&= \int_0^{\mathfrak{T}} \Big[\int_0^{\mathfrak{T}}\aleph_0(\tau,s)\kappa(\tau)\nabla\tau\Big]\phi^{-1}(\varrho(\tau))\nabla\tau + A_2\eta \\
&= \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \Big[\int_0^{\mathfrak{T}}\aleph_0(\tau,s)\kappa(\tau)\nabla\tau\Big]\phi^{-1}(\varrho(\tau))\nabla\tau.
\end{aligned}$$

So, we have

$$\begin{aligned}
\vartheta_1(t) &= -\int_0^{t}(t-\tau)\phi^{-1}(\varrho(\tau))\nabla\tau + \int_0^{\mathfrak{T}}\frac{t}{\mathfrak{T}}(\mathfrak{T}-\tau)\phi^{-1}(\varrho(\tau))\nabla\tau \\
&\quad + \frac{1}{1-\eta}\int_0^{\mathfrak{T}}\Big[\int_0^{\mathfrak{T}}\aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1\Big]\phi^{-1}(\varrho(\tau))\nabla\tau \\
&= \int_0^{t}\frac{\tau}{\mathfrak{T}}(\mathfrak{T}-t)\phi^{-1}(\varrho(\tau))\nabla\tau + \int_t^{\mathfrak{T}}\frac{t}{\mathfrak{T}}(\mathfrak{T}-\tau)\phi^{-1}(\varrho(\tau))\nabla\tau \\
&\quad + \frac{1}{1-\eta}\int_0^{\mathfrak{T}}\Big[\int_0^{\mathfrak{T}}\aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1\Big]\phi^{-1}(\varrho(\tau))\nabla\tau \\
&= \int_0^{\mathfrak{T}}\aleph_0(t,\tau)\phi^{-1}(\varrho(\tau))\nabla\tau \\
&\quad + \frac{1}{1-\eta}\int_0^{\mathfrak{T}}\Big[\int_0^{\mathfrak{T}}\aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1\Big]\phi^{-1}(\varrho(\tau))\nabla\tau \\
&= \int_0^{\mathfrak{T}}\Big[\aleph_0(t,\tau) + \frac{1}{1-\eta}\int_0^{\mathfrak{T}}\aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1\Big]\phi^{-1}(\varrho(\tau))\nabla\tau \\
&= \int_0^{\mathfrak{T}}\aleph(t,\tau)\phi^{-1}(\varrho(\tau))\nabla\tau,
\end{aligned}$$

This completes the proof. □

Lemma 2.6. *Assume that* (H_1) *holds and let* $\mathfrak{z} \in \left(0, \dfrac{\mathfrak{T}}{2}\right)_{\mathbb{T}}$ *and* $\eta_{\mathfrak{z}} = \displaystyle\int_{\mathfrak{z}}^{\mathfrak{T}-\mathfrak{z}} \kappa(t)\nabla t$. *Then* $\aleph_0(t,\tau)$ *and* $\aleph(t,\tau)$ *have the following properties:*

(i) $\aleph_0(t,\tau) > 0$ *and* $\aleph(t,\tau) > 0$ *for all* $t,\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$,

(ii) $\aleph_0(t,\tau) \leq \aleph_0(\tau,\tau),\ \aleph(t,\tau) \leq \aleph(\tau,\tau) \leq \dfrac{1}{1-\eta}\aleph_0(\tau,\tau)$ *for all* $t,\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$,

(iii) $\aleph_0(t,\tau) \geq \dfrac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau)$ *for all* $t \in [\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$

(iv) $\aleph(t,\tau) \geq \lambda_{\mathfrak{z}}\aleph_0(\tau,\tau)$ *where* $\lambda_{\mathfrak{z}} = \dfrac{\mathfrak{z}}{\mathfrak{T}}\left[1 + \dfrac{\eta_{\mathfrak{z}}}{1-\eta}\right]$, *for all* $t \in [\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$.

Proof. Inequalities (i) and (ii) are obvious. To prove (iii), let $t \in [\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}$. Then, for $0 < t < \tau < \mathfrak{T}$,

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{t}{\tau} \geq \frac{\mathfrak{z}}{\mathfrak{T}},$$

and for $0 < \tau < t < \mathfrak{T}$,

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{\mathfrak{T}-t}{\mathfrak{T}-\tau} \geq \frac{\mathfrak{z}}{\mathfrak{T}}.$$

This proves (iii). Next, for $t \in [\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}$, we have

$$\begin{aligned}
\aleph(t,\tau) &= \aleph_0(t,\tau) + \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1 \\
&\geq \frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau) + \frac{1}{1-\eta}\int_{\mathfrak{z}}^{\mathfrak{T}-\mathfrak{z}} \aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1 \\
&\geq \frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau) + \frac{1}{1-\eta}\int_{\mathfrak{z}}^{\mathfrak{T}-\mathfrak{z}} \frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau)\kappa(\tau_1)\nabla\tau_1 \\
&\geq \frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau) + \frac{\eta_{\mathfrak{z}}}{1-\eta}\frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau) \\
&= \lambda_{\mathfrak{z}}\aleph_0(\tau,\tau).
\end{aligned}$$

This completes the proof. □

Notice that an $\ell-$tuple $(\vartheta_1(t),\vartheta_2(t),\vartheta_3(t),\cdots,\vartheta_\ell(t))$ is a solution of the iterative boundary value problem (1.1)–(1.2) if and only if

$$\vartheta_j(t) = \int_0^{\mathfrak{T}} \aleph(t,\tau)\phi^{-1}\big[\zeta(\tau)f_j(\vartheta_{j+1}(\tau))\big]\nabla\tau,\ t \in [0,\mathfrak{T}]_{\mathbb{T}},\ 1 \leq j \leq \ell,$$

$$\vartheta_{\ell+1}(t) = \vartheta_1(t),\ t \in [0,\mathfrak{T}]_{\mathbb{T}},$$

i.e.,

$$\begin{aligned}\vartheta_1(t) = &\int_0^{\mathfrak{T}} \aleph(t,\tau_1)\phi^{-1}\Bigg[\zeta(\tau_1)f_1\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\zeta(\tau_2)f_2\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_2,\tau_3)\\ &\times \phi^{-1}\Bigg[\zeta(\tau_3)f_3\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

Let $\mathtt{X}$ be the Banach space $C_{ld}([0,\mathfrak{T}]_{\mathbb{T}},\mathbb{R})$ with the norm $\|\vartheta\| = \max_{t\in[0,\mathfrak{T}]_{\mathbb{T}}}|\vartheta(t)|$. For $\mathfrak{z}\in\left(0,\frac{\mathfrak{T}}{2}\right)$, we define the cone $\mathtt{P}_{\mathfrak{z}}\subset\mathtt{X}$ as

$$\mathtt{P}_{\mathfrak{z}} = \Big\{\vartheta\in\mathtt{X} : \vartheta(t) \text{ is nonnegative and } \min_{t\in[\mathfrak{z},\,1-\mathfrak{z}]_{\mathbb{T}}}\vartheta(t)\geq\lambda_{\mathfrak{z}}(1-\eta)\|\vartheta(t)\|\Big\},$$

For any $\vartheta_1\in\mathtt{P}_{\mathfrak{z}}$, define an operator $\Omega:\mathtt{P}_{\mathfrak{z}}\to\mathtt{X}$ by

$$\begin{aligned}(\Omega\vartheta_1)(t) = &\int_0^{\mathfrak{T}} \aleph(t,\tau_1)\phi^{-1}\Bigg[\zeta(\tau_1)f_1\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\zeta(\tau_2)f_2\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_2,\tau_3)\\ &\times \phi^{-1}\Bigg[\zeta(\tau_3)f_3\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

Lemma 2.7. *Assume that (H_1)–(H_2) hold. Then for each $\mathfrak{z}\in\left(0,\frac{\mathfrak{T}}{2}\right)$, $\Omega(\mathtt{P}_{\mathfrak{z}})\subset\mathtt{P}_{\mathfrak{z}}$ and $\Omega:\mathtt{P}_{\mathfrak{z}}\to\mathtt{P}_{\mathfrak{z}}$ is completely continuous.*

Proof. From Lemma 2.6, $\aleph(t,\tau)\geq 0$ for all $t,\tau\in[0,\mathfrak{T}]_{\mathbb{T}}$. So, $(\Omega\vartheta_1)(t)\geq 0$. Also, for $\vartheta_1\in\mathtt{P}$, we have

$$\begin{aligned}(\Omega\vartheta_1)(t) \leq &\frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_1,\tau_1)\phi^{-1}\Bigg[\zeta(\tau_1)f_1\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\zeta(\tau_2)\\ &\times f_2\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\zeta(\tau_3)f_3\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

So,

$$\|\Omega\vartheta_1\| \leq \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_1,\tau_1)\phi^{-1}\Bigg[\zeta(\tau_1) f_1\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\zeta(\tau_2)$$
$$\times f_2\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\zeta(\tau_3) f_3\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_3,\tau_4)\cdots$$
$$\times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.$$

Again from Lemma 2.6, we get

$$\min_{t\in[\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\vartheta_1)(t)\} \geq \lambda_{\mathfrak{z}}\int_0^{\mathfrak{T}} \aleph_0(\tau_1,\tau_1)\phi^{-1}\Bigg[\zeta(\tau_1) f_1\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\zeta(\tau_2)$$
$$\times f_2\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\zeta(\tau_3) f_3\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_3,\tau_4)\cdots$$
$$\times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.$$

It follows from the above two inequalities that

$$\min_{t\in[\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\vartheta_1)(t)\} \geq \lambda_{\mathfrak{z}}(1-\eta)\|\Omega\vartheta_1\|.$$

So, $\Omega\vartheta_1 \in \mathtt{P}_{\mathfrak{z}}$ and thus $\Omega(\mathtt{P}_{\mathfrak{z}}) \subset \mathtt{P}_{\mathfrak{z}}$. Next, by standard methods and Arzela–Ascoli theorem, it can be proves easily that the operator Ω is completely continuous. The proof is complete. □

3 Denumerably Infinitely Many Positive Solutions

For the existence of denumerably many positive solutions for iterative system of boundary value problem (1.1). We apply following theorems.

Theorem 3.1. *[10] Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and Λ_1, Λ_2 are open sets with $0 \in \Lambda_1, \overline{\Lambda}_1 \subset \Lambda_2$. Let $\mathcal{A} : \mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1) \to \mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{A}z\| \leq \|z\|,\ z \in \mathcal{E} \cap \partial\Lambda_1$, and $\|\mathcal{A}z\| \geq \|z\|,\ z \in \mathcal{E} \cap \partial\Lambda_2$, or

(b) $\|\mathcal{A}z\| \geq \|z\|,\ z \in \mathcal{E} \cap \partial\Lambda_1$, and $\|\mathcal{A}z\| \leq \|z\|,\ z \in \mathcal{E} \cap \partial\Lambda_2$.

Then $\mathcal{A}$ has a fixed point in $\mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1)$.

Theorem 3.2 (See [7, 17])**.** *Let* $f \in L^p_\nabla(J)$ *with* $p > 1, g \in L^q_\nabla(J)$ *with* $q > 1$*, and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$*. Then* $fg \in L^1_\nabla(J)$ *and* $\|fg\|_{L^1_\nabla} \leq \|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$*.*
where

$$\|f\|_{L^p_\nabla} := \begin{cases} \Big[\displaystyle\int_J |f|^p(s)\nabla s\Big]^{\frac{1}{p}}, & p \in \mathbb{R}, \\ \inf\Big\{M \in \mathbb{R} \,/\, |f| \leq M \ \nabla - a.e., \ on \ J\Big\}, & p = \infty, \end{cases}$$

and $J = (a, b]_{\mathbb{T}}$*.*

Theorem 3.3 (Hölder)**.** *Let* $f \in L^{p_i}_\nabla(J)$ *with* $p_i > 1$*, for* $i = 1, 2, \cdots, n$ *and* $\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} = 1$*. Then* $\displaystyle\prod_{i=1}^{n} f_i \in L^1_\nabla(J)$ *and*

$$\left\|\prod_{i=1}^{n} f_i\right\|_1 \leq \prod_{i=1}^{n} \|f_i\|_{p_i}.$$

Further, if $f \in L^1_\nabla(J)$ *and* $g \in L^\infty_\nabla(J)$*. Then* $fg \in L^1_\nabla(J)$ *and*

$$\|fg\|_1 \leq \|f\|_1\|g\|_\infty.$$

Consider the following three possible cases for $\zeta_i \in L^{p_i}_\nabla([0, \mathfrak{T}]_{\mathbb{T}})$:
$(i) \displaystyle\sum_{i=1}^{n} \frac{1}{p_i} < 1,\ (ii) \sum_{i=1}^{n} \frac{1}{p_i} = 1,\ (iii) \sum_{i=1}^{n} \frac{1}{p_i} > 1.$

Firstly, we seek denumerably many positive solutions for the case $\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} < 1$.

Theorem 3.4. *Suppose* $(H_1)-(H_2)$ *hold, let* $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ *be a sequence with* $\mathfrak{z}_r \in (t_{r+1}, t_r)$*. Let* $\{E_r\}_{r=1}^{\infty}$ *and* $\{O_r\}_{r=1}^{\infty}$ *be such that*

$$E_{r+1} < \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r < O_r < \mathfrak{Z} O_r < E_r,\ r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{\left[\lambda_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau, \tau)\nabla\tau\right]^{-1},\ 1\right\}.$$

Assume that f *satisfies*

(C_1) $f_j(\vartheta) \leq \phi(\mathfrak{M}_1 E_r)\ \forall\, t \in [0, \mathfrak{T}]_{\mathbb{T}},\ 0 \leq \vartheta \leq E_r,$
where

$$\mathfrak{M}_1 < \left[\frac{1}{1-\eta}\|\aleph_0\|_{L^q_\nabla} \prod_{i=1}^{n} \left\|\phi^{-1}(\zeta_i)\right\|_{L^{p_i}_\nabla}\right]^{-1},$$

(C_2) $f_j(\vartheta) \geq \phi(\mathfrak{Z} O_r)\; \forall\, t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}},\ \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r \leq \vartheta \leq O_r.$

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions* $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]}, \cdots, \vartheta_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\vartheta_j^{[r]}(t) \geq 0$ *on* $[0, \mathfrak{T}]_{\mathbb{T}}$, $j = 1, 2, \cdots, \ell$ *and* $r \in \mathbb{N}$.

Proof. Let

$$\Lambda_{1,r} = \{\vartheta \in \mathtt{X} : \|\vartheta\| < E_r\},\ \ \Lambda_{2,r} = \{\vartheta \in \mathtt{X} : \|\vartheta\| < O_r\}$$

be open subsets of $\mathtt{X}$. Let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^* < t_{r+1} < \mathfrak{z}_r < t_r < \frac{\mathfrak{T}}{2},$$

for all $r \in \mathbb{N}$.

For each $r \in \mathbb{N}$, we define the cone $\mathtt{P}_{\mathfrak{z}_r}$ by

$$\mathtt{P}_{\mathfrak{z}_r} = \Big\{\vartheta \in \mathtt{X} : \vartheta(t) \geq 0,\ \min_{t \in [\mathfrak{z}_r,\, \mathfrak{T}-\mathfrak{z}_r]_{\mathbb{T}}} \vartheta(t) \geq \frac{\mathfrak{z}_r}{\mathfrak{T}} \|\vartheta(t)\|\Big\}.$$

Let $\vartheta_1 \in \mathtt{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{1,r}$. Then, $\vartheta_1(\tau) \leq E_r = \|\vartheta_1\|$ for all $\tau \in [0, \mathfrak{T}]_{\mathbb{T}}$. By (C_1) and for $\tau_{\ell-1} \in [0, \mathfrak{T}]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\leq \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)\big]\nabla\tau_\ell \\
&\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}\Big[\prod_{i=1}^{n}\zeta_i(\tau_\ell)\Big]\nabla\tau_\ell \\
&\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell, \tau_\ell)\prod_{i=1}^{n}\phi^{-1}(\zeta_i(\tau_\ell))\nabla\tau_\ell.
\end{aligned}$$

There exists a $q > 1$ such that $\frac{1}{q} + \sum_{i=1}^{n}\frac{1}{p_i} = 1$. So,

$$\begin{aligned}
\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell &\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\Big\|\aleph_0\Big\|_{L^q_\nabla}\Big\|\prod_{i=1}^{n}\phi^{-1}(\zeta_i)\Big\|_{L^{p_i}_\nabla} \\
&\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^{n}\big\|\phi^{-1}(\zeta_i)\big\|_{L^{p_i}_\nabla} \\
&\leq E_r.
\end{aligned}$$

It follows in similar manner (for $\tau_{\ell-2} \in [0, \mathfrak{T}]_{\mathbb{T}},$) that

$$\begin{aligned}
\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-2}, \tau_{\ell-1}) \phi^{-1}\Bigg[\zeta(\tau_{\ell-1}) f_{\ell-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_{\ell}) \phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big] \nabla \tau_\ell\bigg)\Bigg] \nabla \tau_{\ell-1} \\
&\le \int_0^{\mathfrak{T}} \aleph(\tau_{\ell-2}, \tau_{\ell-1}) \phi^{-1}\big[\zeta(\tau_{\ell-1}) f_{\ell-1}(E_r)\big] \nabla \tau_{\ell-1} \\
&\le \frac{1}{1-\eta} \int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1}) \phi^{-1}\big[\zeta(\tau_{\ell-1}) f_{\ell-1}(E_r)\big] \nabla \tau_{\ell-1} \\
&\le \frac{\mathfrak{M}_1 E_r}{1-\eta} \int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1}) \phi^{-1}\big[\zeta(\tau_{\ell-1})\big] \nabla \tau_{\ell-1} \\
&\le \frac{\mathfrak{M}_1 E_r}{1-\eta} \int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1}) \phi^{-1}\left[\prod_{i=1}^{n} \zeta_i(\tau_{\ell-1})\right] \nabla \tau_{\ell-1} \\
&\le \frac{\mathfrak{M}_1 E_r}{1-\eta} \int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1}) \prod_{i=1}^{n} \phi^{-1}(\zeta_i(\tau_{\ell-1})) \nabla \tau_{\ell-1} \\
&\le \frac{\mathfrak{M}_1 E_r}{1-\eta} \|\aleph_0\|_{L^q_\nabla} \prod_{i=1}^{n} \left\|\phi^{-1}(\zeta_i)\right\|_{L^{p_i}_\nabla} \\
&\le E_r.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\Omega\vartheta_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t, \tau_1) \phi^{-1}\Bigg[\zeta(\tau_1) f_1\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1, \tau_2) \phi^{-1}\Bigg[\zeta(\tau_2) f_2\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2, \tau_3) \\
&\quad \times \phi^{-1}\Bigg[\zeta(\tau_3) f_3\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_3, \tau_4) \cdots \\
&\quad \times f_{\ell-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell) \phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big] \nabla \tau_\ell\bigg) \cdots \nabla \tau_3\Bigg] \nabla \tau_2\Bigg] \nabla \tau_1 \\
&\le E_r.
\end{aligned}$$

Since $E_r = \|\vartheta_1\|$ for $\vartheta_1 \in \mathtt{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{1,r}$, we get

$$\|\Omega\vartheta_1\| \le \|\vartheta_1\|. \tag{3.1}$$

Let $t \in [\mathfrak{z}_r, \mathfrak{T} - \mathfrak{z}_r]_{\mathbb{T}}$. Then,

$$O_r = \|\vartheta_1\| \ge \vartheta_1(t) \ge \min_{t \in [\mathfrak{z}_r, \mathfrak{T}-\mathfrak{z}_r]_{\mathbb{T}}} \vartheta_1(t) \ge \frac{\mathfrak{z}_r}{\mathfrak{T}} \|\vartheta_1\| \ge \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r.$$

By (C_2) and for $\tau_{\ell-1} \in [\mathfrak{z}_r, \mathfrak{T} - \mathfrak{z}_r]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\geq \lambda_{\mathfrak{z}_r} \int_{\mathfrak{z}_r}^{\mathfrak{T}-\mathfrak{z}_r} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\geq \lambda_{\mathfrak{z}_r} \mathfrak{Z} O_r \int_{\mathfrak{z}_r}^{\mathfrak{T}-\mathfrak{z}_r} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}(\zeta(\tau_\ell))\nabla\tau_\ell \\
&\geq \lambda_{\mathfrak{z}_r} \mathfrak{Z} O_r \int_{\mathfrak{z}_r}^{\mathfrak{T}-\mathfrak{z}_r} \aleph_0(\tau_\ell, \tau_\ell) \prod_{i=1}^{n} \phi^{-1}(\zeta_i(\tau_\ell))\nabla\tau_\ell \\
&\geq \lambda_{\mathfrak{z}_1} \mathfrak{Z} O_r \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau_\ell, \tau_\ell)\nabla\tau_\ell \\
&\geq O_r.
\end{aligned}$$

Continuing with bootstrapping argument, we get

$$\begin{aligned}
(\Omega\vartheta_1)(t) = &\int_0^{\mathfrak{T}} \aleph(t, \tau_1)\phi^{-1}\Bigg[\zeta(\tau_1) f_1\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1, \tau_2)\phi^{-1}\Bigg[\zeta(\tau_2) f_2\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2, \tau_3) \\
&\times \phi^{-1}\Bigg[\zeta(\tau_3) f_3\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_3, \tau_4) \cdots \\
&\times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg) \cdots \nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1 \\
\geq\, & O_r.
\end{aligned}$$

Thus, if $\vartheta_1 \in \mathtt{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{2,r}$, then

$$\|\Omega\vartheta_1\| \geq \|\vartheta_1\|. \tag{3.2}$$

It is evident that $0 \in \Lambda_{2,k} \subset \overline{\Lambda}_{2,k} \subset \Lambda_{1,k}$. From (3.1),(3.2), it follows from Theorem 3.1 that the operator Ω has a fixed point $\vartheta_1^{[r]} \in \mathtt{P}_{\mathfrak{z}_r} \cap \big(\overline{\Lambda}_{1,r} \backslash \Lambda_{2,r}\big)$ such that $\vartheta_1^{[r]}(t) \geq 0$ on $[0, \mathfrak{T}]_{\mathbb{T}}$, and $r \in \mathbb{N}$. Next setting $\vartheta_{\ell+1} = \vartheta_1$, we obtain denumerably many positive solutions $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]}, \cdots, \vartheta_\ell^{[r]})\}_{r=1}^{\infty}$ of (1.1)–(1.2) given iteratively by

$$\vartheta_j(t) = \int_0^{\mathfrak{T}} \aleph(t, \tau)\phi^{-1}\big[\zeta(\tau) f_j(\vartheta_{j+1}(\tau))\big]\nabla\tau,\ t \in [0, \mathfrak{T}]_{\mathbb{T}},\ j = \ell, \ell-1, \cdots, 1.$$

The proof is completed. □

For $\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} = 1$, we have the following theorem.

Theorem 3.5. *Suppose* $(H_1)-(H_2)$ *hold, let* $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ *be a sequence with* $\mathfrak{z}_r \in (t_{r+1}, t_r)$. *Let* $\{E_r\}_{r=1}^{\infty}$ *and* $\{O_r\}_{r=1}^{\infty}$ *be such that*

$$E_{r+1} < \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r < O_r < \mathfrak{Z} O_r < E_r,\ r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{ \left[\lambda_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau \right]^{-1},\ 1 \right\}.$$

Assume that f *satisfies*

(C_3) $f_j(\vartheta) \le \phi(\mathfrak{M}_2 E_r)\ \forall\, t \in [0,\mathfrak{T}]_{\mathbb{T}},\ 0 \le \vartheta \le E_r,$
where

$$\mathfrak{M}_2 < \min\left\{ \left[\frac{1}{1-\eta} \|\aleph_0\|_{L^{\infty}_{\nabla}} \prod_{i=1}^{n} \left\| \phi^{-1}(\zeta_i) \right\|_{L^{p_i}_{\nabla}} \right]^{-1}, \mathfrak{Z} \right\},$$

(C_4) $f_j(\vartheta) \ge \phi(\mathfrak{Z} O_r)\ \forall\, t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}},\ \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r \le \vartheta \le O_r.$

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions* $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]}, \cdots, \vartheta_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\vartheta_j^{[r]}(t) \ge 0$ *on* $[0,\mathfrak{T}]_{\mathbb{T}}$, $j = 1, 2, \cdots, \ell$ *and* $r \in \mathbb{N}$.

Proof. For a fixed r, let $\Lambda_{1,r}$ be as in the proof of Theorem 3.4 and let $\vartheta_1 \in \mathsf{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{2,r}$. Again

$$\vartheta_1(\tau) \le E_r = \|\vartheta_1\|,$$

for all $\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$. By $(C3)$ and for $\tau_{\ell-1} \in [0,\mathfrak{T}]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\le \frac{1}{1-\eta} \int_0^{\mathfrak{T}} \aleph_0(\tau_\ell,\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta} \int_0^{\mathfrak{T}} \aleph_0(\tau_\ell,\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)\big]\nabla\tau_\ell \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta} \int_0^{\mathfrak{T}} \aleph_0(\tau_\ell,\tau_\ell)\phi^{-1}\left[\prod_{i=1}^{n} \zeta_i(\tau_\ell)\right]\nabla\tau_\ell \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta} \int_0^{\mathfrak{T}} \aleph_0(\tau_\ell,\tau_\ell) \prod_{i=1}^{n} \phi^{-1}(\zeta_i(\tau_\ell))\nabla\tau_\ell \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta} \|\aleph_0\|_{L^{\infty}_{\nabla}} \prod_{i=1}^{n} \left\|\phi^{-1}(\zeta_i)\right\|_{L^{p_i}_{\nabla}} \\
&\le E_r.
\end{aligned}$$

It follows in similar manner (for $\tau_{\ell-2} \in [0, \mathfrak{T}]_{\mathbb{T}},$) that

$$
\begin{aligned}
\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-2}, \tau_{\ell-1}) \phi^{-1}\Big[\zeta(\tau_{\ell-1}) f_{\ell-1}\Big(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Big)\Big]\nabla\tau_{\ell-1} \\
&\le \int_0^{\mathfrak{T}} \aleph(\tau_{\ell-2}, \tau_{\ell-1})\phi^{-1}\big[\zeta(\tau_{\ell-1}) f_{\ell-1}(E_r)\big]\nabla\tau_{\ell-1} \\
&\le \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\phi^{-1}\big[\zeta(\tau_{\ell-1}) f_{\ell-1}(E_r)\big]\nabla\tau_{\ell-1} \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\phi^{-1}\big[\zeta(\tau_{\ell-1})\big]\nabla\tau_{\ell-1} \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\phi^{-1}\Big[\prod_{i=1}^{n}\zeta_i(\tau_{\ell-1})\Big]\nabla\tau_{\ell-1} \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\prod_{i=1}^{n}\phi^{-1}(\zeta_i(\tau_{\ell-1}))\nabla\tau_{\ell-1} \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta}\|\aleph_0\|_{L^\infty_\nabla}\prod_{i=1}^{n}\big\|\phi^{-1}(\zeta_i)\big\|_{L^{p_i}_\nabla} \\
&\le E_r.
\end{aligned}
$$

Continuing with this bootstrapping argument, we get

$$
\begin{aligned}
(\Omega\vartheta_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t, \tau_1)\phi^{-1}\Bigg[\zeta(\tau_1) f_1\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1, \tau_2)\phi^{-1}\Bigg[\zeta(\tau_2) f_2\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2, \tau_3) \\
&\quad \times \phi^{-1}\Bigg[\zeta(\tau_3) f_3\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_3, \tau_4)\cdots \\
&\quad \times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1 \\
&\le E_r.
\end{aligned}
$$

Since $E_r = \|\vartheta_1\|$ for $\vartheta_1 \in \mathtt{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{1,r}$, we get

$$
\|\Omega\vartheta_1\| \le \|\vartheta_1\|. \tag{3.3}
$$

Now define $\Lambda_{2,r} = \{\vartheta_1 \in \mathtt{X} : \|\vartheta_1\| < O_r\}$. Let $\vartheta \in \mathtt{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{2,r}$ and let $\tau \in [\mathfrak{z}_r, \mathfrak{T} - \mathfrak{z}_r]_{\mathbb{T}}$. Then, the argument leading to (3.2) can be done to the present case. Hence, the theorem. □

Lastly, the case $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i} > 1.$

Theorem 3.6. *Suppose* $(H_1)-(H_2)$ *hold, let* $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ *be a sequence with* $\mathfrak{z}_r \in (t_{r+1}, t_r)$. *Let* $\{E_r\}_{r=1}^{\infty}$ *and* $\{O_r\}_{r=1}^{\infty}$ *be such that*

$$E_{r+1} < \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r < O_r < \mathfrak{Z} O_r < E_r, \; r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{ \left[\lambda_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau, \tau) \nabla \tau \right]^{-1}, \; 1 \right\}.$$

Assume that f *satisfies*

(C_5) $f_j(\vartheta) \le \phi(\mathfrak{M}_3 E_r) \; \forall \, t \in [0, \mathfrak{T}]_{\mathbb{T}}, \; 0 \le \vartheta \le E_r,$
where

$$\mathfrak{M}_3 < \min\left\{ \left[\frac{1}{1-\eta} \|\aleph_0\|_{L^{\infty}_{\nabla}} \prod_{i=1}^{n} \left\| \phi^{-1}(\zeta_i) \right\|_{L^1_{\nabla}} \right]^{-1}, \mathfrak{Z} \right\},$$

(C_6) $f_j(\vartheta) \ge \phi(\mathfrak{Z} O_r) \; \forall \, t \in [\mathfrak{z}_r, 1 - \mathfrak{z}_r]_{\mathbb{T}}, \; \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r \le \vartheta \le O_r.$

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions* $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]}, \cdots, \vartheta_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\vartheta_j^{[r]}(t) \ge 0$ *on* $[0, \mathfrak{T}]_{\mathbb{T}}$, $j = 1, 2, \cdots, \ell$ *and* $r \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.1. Therefore, we omit the details here. □

4 Examples

In this section, we present an example to check validity of our main results.

Example 4.1. Consider the following boundary value problem on $\mathbb{T} = [0, 1]$.

$$\left.\begin{array}{c} \phi(\vartheta_j''(t)) + \zeta(t) f_j(\vartheta_{j+1}(t)) = 0, j = 1, 2, \\ \vartheta_3(t) = \vartheta_1(t), \\ \vartheta_j(0) = \vartheta_j(1) = \displaystyle\int_0^1 \frac{1}{2} \vartheta_j(\tau) d\tau \end{array}\right\} \tag{4.1}$$

where

$$\phi(\vartheta) = \begin{cases} \dfrac{\vartheta^3}{1+\vartheta^2}, & \vartheta \le 0, \\ \vartheta^2, & \vartheta > 0, \end{cases}$$

$$\zeta(t) = \zeta_1(t)\zeta_2(t)$$

in which

$$\zeta_1(t) = \frac{1}{|t-\frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \zeta_2(t) = \frac{1}{|t-\frac{1}{3}|^{\frac{1}{2}}},$$

$$f_1(\vartheta) = f_2(\vartheta) = \begin{cases} \frac{1}{50}\times 10^{-16}, & \vartheta \in (10^{-16}, +\infty), \\ \frac{149125\times 10^{-(16r+8)} - \frac{1}{50}\times 10^{-16r}}{10^{-(16r+8)} - 10^{-16r}}(\vartheta - 10^{-16r}) + \frac{1}{50}\times 10^{-16r}, & \vartheta \in \left[10^{-(16r+8)}, 10^{-16r}\right], \\ 149125\times 10^{-(16r+8)}, & \vartheta \in \left(\frac{1}{5}\times 10^{-(16r+8)}, 10^{-(16r+8)}\right), \\ \frac{149125\times 10^{-(16r+8)} - \frac{1}{50}\times 10^{-(16r+16)}}{\frac{1}{5}\times 10^{-(16r+8)} - 10^{-(16r+16)}}(\vartheta - 10^{-(16r+16)}) + \frac{1}{50}\times 10^{-(16r+16)}, & \vartheta \in \left(10^{-(16r+16)}, \frac{1}{5}\times 10^{-(16r+8)}\right]. \end{cases}$$

Let

$$t_r = \frac{31}{64} - \sum_{k=1}^{r} \frac{1}{4(k+1)^4},\ \mathfrak{z}_r = \frac{1}{2}(t_r + t_{r+1}),\ r = 1, 2, 3, \cdots,$$

then

$$\mathfrak{z}_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}$$

and

$$t_{r+1} < \mathfrak{z}_r < t_r,\ \mathfrak{z}_r > \frac{1}{5}.$$

Therefore,

$$\frac{\mathfrak{z}_r}{\mathfrak{T}} = \mathfrak{z}_r > \frac{1}{5},\ j = 1, 2, 3, \cdots.$$

It is clear that

$$t_1 = \frac{15}{32} < \frac{1}{2},\ t_r - t_{r+1} = \frac{1}{4(r+2)^4},\ r = 1, 2, 3, \cdots.$$

Since $\sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}$ and $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{r\to\infty} t_r = \frac{31}{64} - \sum_{k=1}^{\infty} \frac{1}{4(r+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5},$$

$$\zeta_1, \zeta_2 \in L^p[0,1] \quad \text{for all} \quad 0 < p < 2, \quad \text{so} \quad \delta_1 = \delta_2 = \frac{1}{\sqrt{3}}.$$

$$\eta = \int_0^{\mathfrak{T}} \kappa(\tau)\nabla\tau = \int_0^1 \frac{1}{2} d\tau = \frac{1}{2}, \eta_{\mathfrak{z}_1} = \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \kappa(t)\nabla t = \frac{1}{2}(1-2\mathfrak{z}_1) = 0.03279320988,$$

$$\lambda_{\mathfrak{z}_1} = \frac{\mathfrak{z}_1}{\mathfrak{T}}\left[1 + \frac{\eta_{\mathfrak{z}_1}}{1-\eta}\right] = \mathfrak{z}_1(1 + 2\eta_{\mathfrak{z}_1}) = 0.4978492109,$$

$$\int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau = \int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}} \tau(1-\tau)d\tau = 0.01637309451.$$

So, we get

$$\begin{aligned} \mathfrak{Z} &= \max\left\{ \left[\lambda_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau\right]^{-1}, \ 1\right\} \\ &= \max\left\{386.1654402, 1\right\} \\ &= 386.1654402. \end{aligned}$$

$$\|\aleph_0\|_{L^q_\nabla} = \left[\int_0^1 |\aleph_0(\tau,\tau)|^q d\tau\right]^{\frac{1}{q}} < 1 \ \text{ for any } \ 0 < q < 2.$$

Next, let $0 < \mathfrak{a} < 1$ be fixed. Then $\zeta_1, \zeta_2 \in L^{1+\mathfrak{a}}[0,1]$. It follows that

$$\|\phi^{-1}(\zeta_1)\|_{1+\mathfrak{a}} = \left[\frac{1}{3-\mathfrak{a}}\left(3^{\frac{3-\mathfrak{a}}{4}} + 1\right) 2^{\frac{1+\mathfrak{a}}{2}}\right]^{\frac{1}{1+\mathfrak{a}}}$$

$$\|\phi^{-1}(\zeta_2)\|_{1+\mathfrak{a}} = \left[\frac{4}{3-\mathfrak{a}}\left(2^{\frac{3-\mathfrak{a}}{4}} + 1\right)(1/3)^{\frac{3-\mathfrak{a}}{4}}\right]^{\frac{1}{1+\mathfrak{a}}}.$$

So, for $0 < \mathfrak{a} < 1$, we have

$$0.1811770116 \le \left[\frac{1}{1-\eta}\|\aleph_0\|_{L^q_\nabla} \prod_{i=1}^{n} \left\|\phi^{-1}(\zeta_i)\right\|_{L^{p_i}_\nabla}\right]^{-1} \le 185.5612032.$$

Taking $\mathfrak{M}_1 = 0.17$. In addition if we take

$$E_r = 10^{-8r}, \, O_r = 10^{-(8r+4)},$$

then

$$\begin{aligned} E_{r+1} = 10^{-(8r+8)} &< \frac{1}{5} \times 10^{-(8r+4)} < \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r \\ &< O_r = 10^{-(8r+4)} < E_r = 10^{-8r}, \end{aligned}$$

$\mathfrak{Z}O_r = 386.1654402 \times 10^{-(8r+4)} < 0.17 \times 10^{-8r} = \mathfrak{M}_1 E_r$, $r = 1, 2, 3, \cdots$, and f_1, f_2 satisfies the following growth conditions:

$$f_1(\vartheta) = f_2(\vartheta) \leq \phi(\mathfrak{M}_1 E_r) = \mathfrak{M}_1^2 E_r^2 = 0.0289 \times 10^{-16r}, \ \ \vartheta \in \left[0, 10^{-16r}\right]$$

$$\begin{aligned} f_1(\vartheta) = f_2(\vartheta) &\geq \phi(\mathfrak{Z}O_r) = \mathfrak{Z}^2 O_r^2 \\ &= 149123.7162 \times 10^{-(16r+8)}, \ \ \vartheta \in \left[\frac{1}{5} \times 10^{-(16r+8)}, 10^{-(16r+8)}\right]. \end{aligned}$$

Then all the conditions of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the iterative boundary value problem (1.1) has denumerably many solutions $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]})\}_{r=1}^{\infty}$ such that $\vartheta_j^{[r]}(t) \geq 0$ on $[0, 1]$, $j = 1, 2$ and $r \in \mathbb{N}$.

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Infinitely many positive solutions for an iterative system of singular multipoint boundary value problems on time scales

Mahammad Khuddush[1] · **K. Rajendra Prasad**[1] · **K. V. Vidyasagar**[2]

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Abstract
In this paper, we consider an iterative system of singular multipoint boundary value problems on time scales. The sufficient conditions are derived for the existence of infinitely many positive solutions by applying Krasnoselskii's cone fixed point theorem in a Banach space.

Keywords Iterative system · Time scale · Singularity · Cone · Krasnoselskii's fixed point theorem · Positive solutions

Mathematics Subject Classification Primary 34N05 · Secondary 34B18

1 Introduction

Differential equations with state-dependent delays have attracted a great deal of interest to the researchers since they widely arise from application models, such as population models [4], mechanical models [19], infection disease transmission [28], the dynamics of economical systems [5], position control [9], two-body problem of classical electrodynamics [15], etc. As special type of state-dependent delay-differential equations, iterative differential equations have distinctive characteristics and have been investigated in recent years, e.g. equivariance [30], analyticity [31], convexity [27], monotonicity [16], smoothness [12]. Recently [17], Feckan, Wang and Zhao established the maximal and minimal nondecreasing bounded solutions of the following iterative functional differential equations

✉ Mahammad Khuddush
khuddush89@gmail.com

K. Rajendra Prasad
rajendra92@rediffmail.com

K. V. Vidyasagar
vidyavijaya08@gmail.com

1 Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam 530003, India

2 Department of Mathematics, Government Degree College for Women, Marripalem, Koyyuru Mandal, Visakhapatnam 531116, India

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$$x'(t) = g\big(t, x^{(1)}(t), x^{(2)}(t), \dots, x^{(n)}(t)\big),$$

where $x^{(i)}(t) := x(x^{(i-1)})(t)$ indicates the i-th iterate of x, where $i = 1, 2, \dots, n$, by the method of lower and upper solutions.

On the other hand, the theory of time scales was created to unify continuous and discrete analysis. Difference and differential equations can be studied simultaneously by studying dynamic equations on time scales. A time scale is any closed and nonempty subset of the real numbers. So, by this theory, we can extend the continuous and discrete theories to cases "in between." These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Research in this area of mathematics has exceeded by far a thousand publications, and numerous applications to literally all branches of science such as statistics, biology, economics, finance, engineering, physics, and operations research have been given. Moreover, basic results on this issue have been well documented in the articles [1, 2] and monographs of Bohner and Peterson [7, 8]. There is a great deal of research activity devoted to positive solutions of dynamic equations on time scales, see for example [14, 20, 21, 24–26] and references therein.

In [22], Liang and Zhang studied countably many positive solutions for nonlinear singular m–point boundary value problems on time scales,

$$\begin{aligned} &\big(\varphi(x^{\Delta}(t))\big)^{\nabla} + a(t)f\big(x(t)\big) = 0,\ t \in [0, a]_{\mathbb{T}} \\ &x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ x^{\Delta}(a) = 0, \end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [13], Dogan considered second order m–point boundary value problem on time scales,

$$\begin{aligned} &\big(\phi_p(x^{\Delta}(t))\big)^{\nabla} + \omega(t)f\big(t, x(t)\big) = 0,\ t \in [0, T]_{\mathbb{T}} \\ &x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ \phi_p(x^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi_p(x^{\Delta}(\xi_i)), \end{aligned}$$

and established existence of multiple positive solutions by applying fixed-point index theory.

Many researchers have concentrated on studying first order iterative differential equations by different approaches such as fixed point theory, Picard's successive approximation and the technique of nonexpansive operators. But the literature related to the equations of higher order is limited since the presence of the iterates increases the difficulty of studying them. This motivates us to investigate the following second order dynamical iterative system of boundary value problems with singularities on time scales,

$$\left.\begin{aligned} &x_{\ell}^{\Delta\nabla}(t) + \lambda(t) g_{\ell}\big(x_{\ell+1}(t)\big) = 0,\ 1 \le \ell \le n,\ t \in (0, \sigma(a)]_{\mathbb{T}} \\ &x_{n+1}(t) = x_1(t),\ t \in (0, \sigma(a)]_{\mathbb{T}}, \end{aligned}\right\} \tag{1}$$

$$x_{\ell}^{\Delta}(0) = 0,\ x_{\ell}(\sigma(a)) = \sum_{k=1}^{n-2} c_k x_{\ell}(\zeta_k),\ 1 \le \ell \le n, \tag{2}$$

where $n \in \mathbb{N}$, $c_k \in \mathbb{R}^+ := [0, +\infty)$ with $\sum_{k=1}^{n-2} c_k < 1$, $0 < \zeta_k < \sigma(a)/2$, $k \in \{1, 2, \dots, n-2, \}$, $\lambda(t) = \prod_{i=1}^{m} \lambda_i(t)$ and each $\lambda_i(t) \in L_\nabla^{p_i}((0, \sigma(a)]_\mathbb{T})(p_i \geq 1)$ has a singularity in the interval $(0, \sigma(a)/2]_\mathbb{T}$. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of infinitely many positive solutions for the system (1). Equation (1) in real continuous time scales describes diffusion phenomena with a source or a reaction term. For instance, in thermal conduction, it can be interpreted as the one-dimensional heat conduction equation which models the steady-states of a heated bar of length a with a controller at $x = a$ that adds or removes heat according to a sensor, while the left endpoint is maintained at 0°C and g is the distributed temperature source function depending on delayed temperatures. We refer the interested reader to [10, 11] and the references therein for more details.

We assume the following conditions are true throughout the paper:

($\mathcal{H}_1$) $g_\ell : [0, +\infty) \to [0, +\infty)$ is continuous.
($\mathcal{H}_2$) there exists a sequence $\{t_r\}_{r=1}^{\infty}$ such that $0 < t_{r+1} < t_r < \sigma(a)/2$,

$$\lim_{r\to\infty} t_r = t^* < \sigma(a)/2, \ \lim_{t\to t_r} \lambda_i(t) = +\infty, \ i = 1, 2, \dots, m.$$

Further, for each $i \in \{1, 2, \dots, m\}$, there exist $\delta_i > 0$ such that $\lambda_i(t) > \delta_i$.

2 Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions.

Definition 2.1 [7] A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to [0, +\infty)$ are defined by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$, and $\mu(t) = \sigma(t) - t$, respectively.

- The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t, \sigma(t) = t, \sigma(t) > t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$.
- If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.
- By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e., $[a, b]_\mathbb{T} = [a, b] \cap \mathbb{T}$. Other intervals can be defined similarly.

Definition 2.2 [6] Let μ_Δ and μ_∇ be the Lebesgue $\Delta-$ measure and the Lebesgue $\nabla-$measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.

(i) If there exists $\Gamma_1 \subset A$ with $\mu_\Delta(\Gamma_1) = 0$ such that P holds on $A \backslash \Gamma_1$, then P is said to hold Δ–a.e. on A.
(ii) If there exists $\Gamma_2 \subset A$ with $\mu_\nabla(\Gamma_2) = 0$ such that P holds on $A \backslash \Gamma_2$, then P is said to hold ∇–a.e. on A.

Definition 2.3 [3, 6] Let $E \subset \mathbb{T}$ be a Δ–measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be Δ–measurable function. We say that f belongs to $L^p_\Delta(E)$ provided that either

$$\int_E |f|^p(s)\Delta s < \infty \quad \text{if} \quad p \in [1, +\infty),$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M, \quad \Delta - a.e.\ on\ E \ \text{ if } \ p = +\infty.$$

Lemma 2.4 [29] *Let $E \subset \mathbb{T}$ be a Δ–measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is Δ–integrable on E, then*

$$\int_E f(s)\Delta s = \int_E f(s)ds + \sum_{i \in I_E} \big(\sigma(t_i) - t_i\big) f(t_i) + r(f, E),$$

where

$$r(f, E) = \begin{cases} \mu_{\mathbb{N}}(E) f(M), & \text{if } \mathbb{N} \in \mathbb{T}, \\ 0, & \text{if } \mathbb{N} \notin \mathbb{T}, \end{cases}$$

$I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}, I \subset \mathbb{N}$, is the set of all right-scattered points of $\mathbb{T}$.

Definition 2.5 [29] Let $E \subset \mathbb{T}$ be a ∇–measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be ∇–measurable function. Say that f belongs to $L^p_\nabla(E)$ provided that either

$$\int_E |f|^p(s)\nabla s < \infty \quad \text{if} \quad p \in \mathbb{R},$$

or there exists a constant $C \in \mathbb{R}$ such that

$$|f| \leq C, \quad \nabla - a.e.\ on\ E \ \text{ if } \ p = +\infty.$$

Lemma 2.6 [29] *Let $E \subset \mathbb{T}$ be a ∇–measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is a ∇–integrable on E, then*

$$\int_E f(s)\nabla s = \int_E f(s)ds + \sum_{i \in I_E} \big(t_i - \rho(t_i)\big) f(t_i),$$

where $I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}, I \subset \mathbb{N}$, is the set of all left-scattered points of $\mathbb{T}$.

Lemma 2.7 *For any $\mathrm{y}(t) \in \mathcal{C}_{ld}((0, \sigma(a)]_\mathbb{T})$, the boundary value problem,*

$$\mathrm{x}_1^{\Delta\nabla}(t) + \mathrm{y}(t) = 0, \ t \in (0, \sigma(a)]_\mathbb{T}, \tag{3}$$

$$\mathtt{x}_1^{\Delta}(0) = 0,\ \mathtt{x}_1(\sigma(a)) = \sum_{k=1}^{n-2} c_k \mathtt{x}_1(\zeta_k) \tag{4}$$

has a unique solution

$$\mathtt{x}_1(t) = \int_0^{\sigma(a)} \aleph(t,\tau)\mathtt{y}(\tau)\nabla\tau + \frac{1}{1-\sum_{k=1}^{n-2} c_k}\sum_{k=1}^{n-2} c_k \int_0^{\sigma(a)} \aleph(\zeta_k,\tau)\mathtt{y}(\tau)\nabla\tau, \tag{5}$$

where

$$\aleph(t,\tau) = \begin{cases} \sigma(a)-t, & \text{if } 0 \le \tau \le t \le \sigma(a), \\ \sigma(a)-\tau, & \text{if } 0 \le t \le \tau \le \sigma(a). \end{cases} \tag{6}$$

Proof Suppose $\mathtt{x}_1$ is a solution of (3), then

$$\begin{aligned} \mathtt{x}_1(t) &= -\int_0^t \int_0^\tau \mathtt{y}(\tau_1)\nabla\tau_1\Delta\tau + \mathtt{A}t + \mathtt{B} \\ &= -\int_0^t (t-\tau)\mathtt{y}(\tau)\nabla\tau + \mathtt{A}t + \mathtt{B}, \end{aligned}$$

where $\mathtt{A} = \mathtt{x}_1^{\Delta}(0)$ and $\mathtt{X} = \mathtt{x}_1(0)$. Using conditions (4), we get $\mathtt{A} = 0$ and

$$\mathtt{B} = \int_0^{\sigma(a)} (\sigma(a)-\tau)\mathtt{y}(\tau)\nabla\tau + \sum_{k=1}^{n-2} c_k \mathtt{x}_1(\zeta_k).$$

So, we have

$$\begin{aligned} \mathtt{x}_1(t) &= -\int_0^t (t-\tau)\mathtt{y}(\tau)\nabla\tau + \int_0^{\sigma(a)} (\sigma(a)-\tau)\mathtt{y}(\tau)\nabla\tau + \sum_{k=1}^{n-2} c_k \mathtt{x}_1(\zeta_k) \\ &= \int_0^{\sigma(a)} \aleph(t,\tau)\mathtt{y}(\tau)\nabla\tau + \sum_{k=1}^{n-2} c_k \mathtt{x}_1(\zeta_k). \end{aligned} \tag{7}$$

Plugging $t = \zeta_k$ and multiplying with c_k then summing from 1 to $n-2$ in the above equation (7), we obtain

$$\mathtt{x}_1(\zeta_k) = \frac{1}{1-\sum_{k=1}^{n-2} c_k}\sum_{k=1}^{n-2} c_k \int_0^{\sigma(a)} \aleph(\zeta_k,\tau)\mathtt{y}(\tau)\nabla\tau. \tag{8}$$

Substituting (8) into (7), we get required solution (5). This completes the proof. □

Lemma 2.8 *Suppose* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *hold. Let* $\eta \in (0, \sigma(a)/2)_{\mathbb{T}}$ *with* $\zeta_k \in [\eta, \sigma(a)-\eta]_{\mathbb{T}}$, $k \in \{1,2,\cdots,n-2\}$, *the kernel* $\aleph(t,\tau)$ *have the following properties*:

(i) $0 \le \aleph(t,\tau) \le \aleph(\tau,\tau)$ *for all* $t,\tau \in [0,\sigma(a)]_{\mathbb{T}}$,
(ii) $\frac{\eta}{\sigma(a)}\aleph(\tau,\tau) \le \aleph(t,\tau)$ *for all* $t \in [\eta,\sigma(a)-\eta]_{\mathbb{T}}$ *and* $\tau \in [0,\sigma(a)]_{\mathbb{T}}$.

Proof (i) is evident. To prove (ii), let $t \in [\eta,\sigma(a)-\eta]_{\mathbb{T}}$ and $\tau \le t$. Then

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)}=\frac{\sigma(a)-t}{\sigma(a)-\tau}\geq\frac{\eta}{\sigma(a)}.$$

For $t\leq\tau$,

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)}=\frac{\sigma(a)-\tau}{\sigma(a)-\tau}=1\geq\frac{\eta}{\sigma(a)}.$$

This completes the proof. □

Notice that an n–tuple $(\mathrm{x}_1(t),\mathrm{x}_2(t),\mathrm{x}_3(t),\ldots,\mathrm{x}_n(t))$ is a solution of the iterative boundary value problem (1)–(2) if and only if

$$\begin{aligned}\mathrm{x}_\ell(t)=&\int_0^{\sigma(a)}\aleph(t,\tau)\lambda(\tau)\mathrm{g}_\ell(\mathrm{x}_{\ell+1}(\tau))\nabla\tau\\&+\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\zeta_k,\tau)\lambda(\tau)\mathrm{g}_\ell(\mathrm{x}_{\ell+1}(\tau))\nabla\tau\end{aligned}$$

and

$$\mathrm{x}_{\ell+1}(t)=\mathrm{x}_1(t),\ t\in(0,a]_{\mathbb{T}},\ 1\leq\ell\leq n.$$

That is

$$\begin{aligned}\mathrm{x}_1(t)=&\int_0^{\sigma(a)}\aleph(t,\tau_1)\lambda(\tau_1)\mathrm{g}_1\Bigg[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathrm{g}_2\Bigg[\int_0^{\sigma(a)}\aleph(\tau_2,\tau_3)\cdots\\&\times\mathrm{g}_{n-1}\Bigg[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\Delta\tau_n\Bigg]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1\\&+\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\zeta_k,\tau_1)\lambda(\tau_1)\mathrm{g}_1\Bigg[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\\&\times\mathrm{g}_{n-1}\Bigg[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\Delta\tau_n\Bigg]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1.\end{aligned}$$

Let $\mathtt{X}$ be the Banach space $C_{ld}((0,\sigma(a)]_{\mathbb{T}},\mathbb{R})$ with the norm $\|\mathrm{x}\|=\max_{t\in(0,\sigma(a)]_{\mathbb{T}}}|\mathrm{x}(t)|$. For $\eta\in(0,\sigma(a)/2)_{\mathbb{T}}$, we define the cone $\mathtt{P}_\eta\subset\mathtt{X}$ as

$$\mathtt{P}_\eta=\left\{\mathtt{x}\in\mathtt{X}:\mathrm{x}(t)\text{ is nonnegative and }\min_{t\in[\eta,\sigma(a)-\eta]_{\mathbb{T}}}\mathrm{x}(t)\geq\frac{\eta}{\sigma(a)}\|\mathrm{x}(t)\|\right\},$$

For any $\mathtt{x}_1\in\mathtt{P}_\eta$, define an operator $\mathscr{L}:\mathtt{P}_\eta\to\mathtt{X}$ by

$$(\mathscr{L}x_1)(t) = \int_0^{\sigma(a)} \aleph(t,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)g_2\left[\int_0^{\sigma(a)} \aleph(\tau_2,\tau_3)\cdots\right.\right.$$
$$\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1$$
$$+\frac{1}{1-\sum_{k=1}^{n-2} c_k}\sum_{k=1}^{n-2} c_k \int_0^{\sigma(a)} \aleph(\zeta_k,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1.$$

Lemma 2.9 *Assume that* $(\mathcal{H}_1)$*–*$(\mathcal{H}_2)$ *hold. Then for each* $\eta \in (0,\sigma(a)/2)_{\mathbb{T}}$, $\mathscr{L}(P_\eta) \subset P_\eta$ *and* $\mathscr{L}: P_\eta \to P_\eta$ *are completely continuous.*

Proof From Lemma 2.8, $\aleph(t,\tau) \ge 0$ for all $t,\tau \in (0,\sigma(a)]_{\mathbb{T}}$. So, $(\mathscr{L}x_1)(t) \ge 0$. Also, for $x_1 \in P_\eta$, we have

$$\|\mathscr{L}x_1\| = \max_{t\in(0,\sigma(a)]_{\mathbb{T}}} \int_0^{\sigma(a)} \aleph(t,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1$$
$$+\frac{1}{1-\sum_{k=1}^{n-2} c_k}\sum_{k=1}^{n-2} c_k \int_0^{\sigma(a)} \aleph(\zeta_k,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1$$
$$\le \int_0^{\sigma(a)} \aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1$$
$$+\frac{1}{1-\sum_{k=1}^{n-2} c_k}\sum_{k=1}^{n-2} c_k \int_0^{\sigma(a)} \aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\right]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1.$$

Again from Lemma 2.8, we get

$$\min_{t\in[\eta,a-\eta]_{\mathbb{T}}}\left\{(\mathscr{L}x_1)(t)\right\} \geq \frac{\eta}{\sigma(a)}\Bigg[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1\Bigg[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots$$
$$\times g_{n-1}\Bigg[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\Bigg]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1$$
$$+\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1\Bigg[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots$$
$$\times g_{n-1}\Bigg[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n\Bigg]\cdots\Delta\tau_3\Bigg]\Delta\tau_2\Bigg]\Delta\tau_1\Bigg].$$

It follows from the above two inequalities that

$$\min_{t\in[\eta,a-\eta]_{\mathbb{T}}}\left\{(\mathscr{L}x_1)(t)\right\} \geq \frac{\eta}{\sigma(a)}\|\mathscr{L}x_1\|.$$

So, $\mathscr{L}x_1 \in \mathsf{P}_\eta$ and thus $\mathscr{L}(\mathsf{P}_\eta) \subset \mathsf{P}_\eta$. Next, by standard methods and Arzela-Ascoli theorem, it can be proved easily that the operator $\mathscr{L}$ is completely continuous. The proof is complete. □

3 Infinitely many positive solutions

For the the existence of infinitely many positive solutions for iterative system of boundary value problem (1)–(2). We apply following theorems.

Theorem 3.1 **(Krasnoselskii's** [18]) *Let $\mathcal{B}$ be a cone in a Banach space $\mathcal{E}$ and Q_1, Q_2 are open sets with $0 \in \mathsf{Q}_1, \overline{\mathsf{Q}}_1 \subset \mathsf{Q}_2$. Let $\mathcal{K} : \mathcal{B}\cap(\overline{\mathsf{Q}}_2\backslash\mathsf{Q}_1) \to \mathcal{B}$ be a completely continuous operator such that*

(a) $\|\mathcal{K}v\| \leq \|v\|,\ v \in \mathcal{B}\cap\partial\mathsf{Q}_1$, and $\|\mathcal{K}v\| \geq \|v\|,\ v\in\mathcal{B}\cap\partial\mathsf{Q}_2$, or
(b) $\|\mathcal{K}v\| \geq \|v\|,\ v \in \mathcal{B}\cap\partial\mathsf{Q}_1$, and $\|\mathcal{K}v\| \leq \|v\|,\ v\in\mathcal{B}\cap\partial\mathsf{Q}_2$.

Then $\mathcal{K}$ has a fixed point in $\mathcal{B}\cap(\overline{\mathsf{Q}}_2\backslash\mathsf{Q}_1)$.

Theorem 3.2 **(Hölder's Inequality** [3, 23]) *Let $f \in L^p_\nabla(I)$ with $p>1$, $g\in L^q_\nabla(I)$ with $q>1$, and $\frac{1}{p}+\frac{1}{q}=1$. Then $fg\in L^1_\nabla(I)$ and $\|fg\|_{L^1_\nabla} \leq \|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$. where*

$$\|f\|_{L^p_\nabla} := \begin{cases}\left[\int_I |f|^p(s)\nabla s\right]^{\frac{1}{p}}, p\in\mathbb{R},\\ \inf\left\{K\in\mathbb{R}\,/\,|f|\leq K\ \nabla-a.e.,\ on\, I\right\},\quad p=\infty,\end{cases}$$

and $I=[a,b]_{\mathbb{T}}$. Moreover, if $f\in L^1_\nabla(I)$ and $g\in L^\infty_\nabla(I)$. Then $fg\in L^1_\nabla(I)$ and $\|fg\|_{L^1_\nabla}\leq\|f\|_{L^1_\nabla}\|g\|_{L^\infty_\nabla}$.

Consider the following three possible cases for $\lambda_i \in L^{p_i}_\Delta(0,\sigma(a)]_{\mathbb{T}}$:

$$\sum_{i=1}^{m}\frac{1}{p_i}<1,\ \sum_{i=1}^{m}\frac{1}{p_i}=1,\ \sum_{i=1}^{m}\frac{1}{p_i}>1.$$

Firstly, we seek infinitely many positive solutions for the case $\sum_{i=1}^{m}\frac{1}{p_i}<1$.

Theorem 3.3 *Suppose* $(\mathcal{H}_1)$*-*$(\mathcal{H}_2)$ *hold, let* $\{\eta_r\}_{r=1}^{\infty}$ *be a sequence with* $t_{r+1}<\eta_r<t_r$. *Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Lambda_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1}<\frac{\eta_r}{\sigma(a)}\Lambda_r<\Lambda_r<\theta\Lambda_r<\Gamma_r \text{ and } \frac{\eta_r}{\sigma(a)}<\frac{1}{2},\ r\in\mathbb{N},$$

where

$$\theta=\max\left\{\left[\frac{\eta_1}{\sigma(a)}\prod_{i=1}^{m}\delta_i\int_{\eta_1}^{\sigma(a)-\eta_1}\aleph(\tau,\tau)\Delta\tau\right]^{-1},\right.$$
$$\left.\left[\frac{\sum_{k=1}^{n-2}c_k}{1-\sum_{k=1}^{n-2}c_k}\frac{\eta_1}{\sigma(a)}\prod_{i=1}^{m}\delta_i\int_{\eta_1}^{\sigma(a)-\eta_1}\aleph(\tau,\tau)\nabla\tau\right]^{-1}\right\}.$$

Assume that g_ℓ *satisfies*

(J_1) $g_\ell(x)\le\frac{\mathfrak{N}_1\Gamma_r}{2}\ \forall\, t\in(0,\sigma(a)]_{\mathbb{T}},\ 0\le x\le\Gamma_r$, where

$$\mathfrak{N}_1<\min\left\{\left[\|\aleph\|_{L^q_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla}\right]^{-1},\ \left[\frac{\sum_{k=1}^{n-2}c_k}{1-\sum_{k=1}^{n-2}c_k}\|\aleph\|_{L^q_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla}\right]^{-1}\right\},$$

(J_2) $g_\ell(x)\ge\frac{\theta\Lambda_r}{2}\ \forall\, t\in[\eta_r,\sigma(a)-\eta_r]_{\mathbb{T}},\ \frac{\eta_r}{\sigma(a)}\Lambda_r\le x\le\Lambda_r$.

Then the iterative boundary value problem (1)–(2) has infinitely many solutions $\{(x_1^{[r]},x_2^{[r]},\cdots,x_n^{[r]})\}_{r=1}^{\infty}$ such that $x_\ell^{[r]}(t)\ge0$ on $(0,\sigma(a)]_{\mathbb{T}}$, $\ell=1,2,\cdots,n$ and $r\in\mathbb{N}$.

Proof Let

$$Q_{1,r}=\{x\in X:\|x\|<\Gamma_r\},\ Q_{2,r}=\{x\in X:\|x\|<\Lambda_r\}$$

be open subsets of X. Let $\{\eta_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^*<t_{r+1}<\eta_r<t_r<\frac{\sigma(a)}{2},$$

for all $r\in\mathbb{N}$. For each $r\in\mathbb{N}$, we define the cone P_{η_r} by

$$P_{\eta_r}=\left\{x\in X: x(t)\ge0,\ \min_{t\in[\eta_r,\sigma(a)-\eta_r]_{\mathbb{T}}}x(t)\ge\frac{\eta_r}{\sigma(a)}\|x(t)\|\right\}.$$

Let $x_1\in P_{\eta_r}\cap\partial Q_{1,r}$. Then, $x_1(\tau)\le\Gamma_r=\|x_1\|$ for all $\tau\in(0,\sigma(a)]_{\mathbb{T}}$. By (J_1) and for $\tau_{m-1}\in(0,\sigma(a)]_{\mathbb{T}}$, we have

$$\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n \le \int_0^{\sigma(a)} \aleph(\tau_n, \tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n$$
$$\le \frac{\mathfrak{N}_1\Gamma_r}{2} \int_0^{\sigma(a)} \aleph(\tau_n, \tau_n) \prod_{i=1}^{m} \lambda_i(\tau_n)\nabla\tau_n.$$

There exists a $q > 1$ such that $\frac{1}{q} + \sum_{i=1}^{n} \frac{1}{p_i} = 1$. So,

$$\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\Delta\tau_n \le \frac{\mathfrak{N}_1\Gamma_r}{2} \|\aleph\|_{L^q_\nabla} \left\| \prod_{i=1}^{m} \lambda_i \right\|_{L^{p_i}_\nabla}$$
$$\le \frac{\mathfrak{N}_1\Gamma_r}{2} \|\aleph\|_{L^q_\nabla} \prod_{i=1}^{m} \|\lambda_i\|_{L^{p_i}_\nabla} \le \frac{\Gamma_r}{2} < \Gamma_r.$$

It follows in similar manner (for $\tau_{n-2} \in (0, \sigma(a)]_{\mathbb{T}}$,) that

$$\int_0^{\sigma(a)} \aleph(\tau_{n-2}, \tau_{n-1})\lambda(\tau_{n-1})g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n\right]\nabla\tau_{n-1}$$
$$\le \int_0^{\sigma(a)} \aleph(\tau_{n-2}, \tau_{n-1})\lambda(\tau_{n-1})g_{n-1}(\Gamma_r)\nabla\tau_{n-1}$$
$$\le \int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_{n-1})\lambda(\tau_{n-1})g_{n-1}(\Gamma_r)\nabla\tau_{n-1}$$
$$\le \frac{\mathfrak{N}_1\Gamma_r}{2} \int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_{n-1}) \prod_{i=1}^{m} \lambda_i(\tau_{n-1})\nabla\tau_{n-1}$$
$$\le \frac{\mathfrak{N}_1\Gamma_r}{2} \|\aleph\|_{L^q_\nabla} \prod_{i=1}^{m} \|\lambda_i\|_{L^{p_i}_\nabla} \le \frac{\Gamma_r}{2} < \Gamma_r.$$

Continuing with this bootstrapping argument, we get

$$\int_0^{\sigma(a)} \aleph(t, \tau_1)\lambda(\tau_1)g_1\left[\int_0^{\sigma(a)} \aleph(\tau_1, \tau_2)\lambda(\tau_2)g_2\left[\int_0^{\sigma(a)} \aleph(\tau_2, \tau_3) \cdots\right.\right.$$
$$\left.\left.\times g_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n\right] \cdots \nabla\tau_3\right]\nabla\tau_2\right]\nabla\tau_1 \le \frac{\Gamma_r}{2}.$$

Also, we note that

$$\frac{1}{1-\sum_{k=1}^{n-2} c_k} \sum_{k=1}^{n-2} c_k \int_0^{\sigma(a)} \aleph(\zeta_k, \tau_1)\lambda(\tau_1)\mathrm{g}_1\left[\int_0^{\sigma(a)} \aleph(\tau_1, \tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times\, \mathrm{g}_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\nabla\tau_n\right]\cdots\nabla\tau_2\right]\nabla\tau_1$$
$$\leq \frac{1}{1-\sum_{k=1}^{n-2} c_k} \sum_{k=1}^{n-2} c_k \int_0^{\sigma(a)} \aleph(\tau_1, \tau_1)\lambda(\tau_1)\mathrm{g}_1(\Gamma_r)\nabla\tau_1$$
$$\leq \frac{1}{1-\sum_{k=1}^{n-2} c_k} \sum_{k=1}^{n-2} c_k \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^q_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla} \leq \frac{\Gamma_r}{2}.$$

Thus, $(\mathscr{L}\mathrm{x}_1)(t) \leq \frac{\Gamma_r}{2} + \frac{\Gamma_r}{2} = \Gamma_r$. Since $\Gamma_r = \|\mathrm{x}_1\|$ for $\mathrm{x}_1 \in \mathtt{P}_{\eta_r} \cap \partial\mathtt{Q}_{1,r}$, we get

$$\|\mathscr{L}\mathrm{x}_1\| \leq \|\mathrm{x}_1\|. \tag{9}$$

Next, let $t \in [\eta_r, \sigma(a)-\eta_r]_{\mathbb{T}}$. Then,

$$\Lambda_r = \|\mathrm{x}_1\| \geq \mathrm{x}_1(t) \geq \min_{t\in[\eta_r, a-\eta_r]_{\mathbb{T}}} \mathrm{x}_1(t) \geq \frac{\eta_r}{\sigma(a)}\|\mathrm{x}_1\| \geq \frac{\eta_r}{\sigma(a)}\Lambda_r.$$

By (J_2) and for $\tau_{n-1} \in [\eta_r, \sigma(a)-\eta_r]_{\mathbb{T}}$, we have

$$\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\big]\nabla\tau_n$$
$$\geq \int_{\eta_r}^{\sigma(a)-\eta_r} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\nabla\tau_n$$
$$\geq \frac{\eta_r}{\sigma(a)}\frac{\theta\Lambda_r}{2}\int_{\eta_r}^{\sigma(a)-\eta_r} \aleph(\tau_n, \tau_n)\lambda(\tau_n))\nabla\tau_n$$
$$\geq \frac{\eta_r}{\sigma(a)}\frac{\theta\Lambda_r}{2}\int_{\eta_r}^{\sigma(a)-\eta_r} \aleph(\tau_n, \tau_n)\prod_{i=1}^{m}\lambda_i(\tau_n))\nabla\tau_n$$
$$\geq \frac{\eta_1}{\sigma(a)}\frac{\theta\Lambda_r}{2}\prod_{i=1}^{m}\delta_i\int_{\eta_1}^{\sigma(a)-\eta_1} \aleph(\tau_n, \tau_n)\nabla\tau_n$$
$$\geq \frac{\Lambda_r}{2}.$$

and

$$\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\zeta_k,\tau_1)\lambda(\tau_1)g_1\Bigg[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots$$
$$\times g_{n-1}\Bigg[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)g_n(x_1(\tau_n))\nabla\tau_n\Bigg]\cdots\nabla\tau_2\Bigg]\nabla\tau_1$$
$$\geq\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\frac{\eta_1}{\sigma(a)}\int_0^{\sigma(a)}\aleph(\tau_1,\tau_1)\lambda(\tau_1)g_1(\Gamma_r)\nabla\tau_1$$
$$\geq\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\frac{\eta_1}{\sigma(a)}\frac{\theta\Lambda_r}{2}\prod_{i=1}^{m}\delta_i\int_{\eta_1}^{\sigma(a)-\eta_1}\aleph(\tau_1,\tau_1)\nabla\tau_1$$

Continuing with bootstrapping argument, we get $(\mathscr{L}x_1)(t)\geq\frac{\Lambda_r}{2}+\frac{\Lambda_r}{2}=\Lambda_r$. Thus, if $x_1\in P_{\eta_r}\cap\partial P_{2,r}$, then

$$\|\mathscr{L}x_1\|\geq\|x_1\|. \tag{10}$$

It is evident that $0\in Q_{2,k}\subset\overline{Q}_{2,k}\subset Q_{1,k}$. From (9),(10), it follows from Theorem 3.1 that the operator $\mathscr{L}$ has a fixed point $x_1^{[r]}\in P_{\eta_r}\cap\left(\overline{Q}_{1,r}\backslash Q_{2,r}\right)$ such that $x_1^{[r]}(t)\geq 0$ on $(0,a]_{\mathbb{T}}$, and $r\in\mathbb{N}$. Next setting $x_{m+1}=x_1$, we obtain infinitely many positive solutions $\{(x_1^{[r]},x_2^{[r]},\ldots,x_m^{[r]})\}_{r=1}^{\infty}$ of (1)–(2) given iteratively by

$$x_\ell(t)=\int_0^{\sigma(a)}\aleph(t,\tau)\lambda(\tau)g_\ell(x_{\ell+1}(\tau))\nabla\tau,\ t\in(0,\sigma(a)]_{\mathbb{T}},\ \ell=n,n-1,\ldots,1.$$

The proof is completed. □

For $\sum_{i=1}^{m}\frac{1}{p_i}=1$, we have the following theorem.

Theorem 3.4 *Suppose* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *hold, let* $\{\eta_r\}_{r=1}^{\infty}$ *be a sequence with* $t_{r+1}<\eta_r<t_r$. *Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Lambda_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1}<\frac{\eta_r}{\sigma(a)}\Lambda_r<\Lambda_r<\theta\Lambda_r<\Gamma_r\quad\text{and}\quad\frac{\eta_r}{\sigma(a)}<\frac{1}{2},\ r\in\mathbb{N}.$$

Assume that g_ℓ *satisfies* (J_2) *and*

(J_3) $g_\ell(x)\leq\frac{\mathfrak{N}_2\Gamma_r}{2}\,\forall\, t\in(0,\sigma(a)]_{\mathbb{T}},\ 0\leq x\leq\Gamma_r$, where

$$\mathfrak{N}_2<\min\left\{\left[\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla}\right]^{-1},\left[\frac{\sum_{k=1}^{n-2}c_k}{1-\sum_{k=1}^{n-2}c_k}\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla}\right]^{-1}\right\}.$$

Then the iterative boundary value problem (1)–(2) has infinitely many solutions $\left\{\left(x_1^{[r]},x_2^{[r]},\ldots,x_n^{[r]}\right)\right\}_{r=1}^{\infty}$ such that $x_\ell^{[r]}(t)\geq 0$ on $(0,\sigma(a)]_{\mathbb{T}}$, $\ell=1,2,\ldots,n$ and $r\in\mathbb{N}$.

Proof For a fixed r, let $Q_{1,r}$ be as in the proof of Theorem 3.3 and let $x_1\in P_{\eta_r}\cap\partial Q_{2,r}$. Again

$$\mathrm{x}_1(\tau) \leq \Gamma_r = \|\mathrm{x}_1\|,$$

for all $\tau \in (0, \sigma(a)]_{\mathbb{T}}$. By (J_3) and for $\tau_{\ell-1} \in (0, \sigma(a)]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\nabla\tau_n &\leq \int_0^{\sigma(a)} \aleph(\tau_n, \tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\nabla\tau_n \\ &\leq \frac{\mathfrak{N}_1\Gamma_r}{2}\int_0^{\sigma(a)} \aleph(\tau_n, \tau_n)\prod_{i=1}^m \lambda_i(\tau_n)\nabla\tau_n \\ &\leq \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^\infty_\nabla}\left\|\prod_{i=1}^m \lambda_i\right\|_{L^{p_i}_\nabla} \\ &\leq \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^m \|\lambda_i\|_{L^{p_i}_\nabla} \leq \frac{\Gamma_r}{2} < \Gamma_r.\end{aligned}$$

It follows in similar manner (for $\tau_{n-2} \in (0, \sigma(a)]_{\mathbb{T}}$,) that

$$\begin{aligned}&\int_0^{\sigma(a)} \aleph(\tau_{n-2}, \tau_{n-1})\lambda(\tau_{n-1})\mathrm{g}_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\nabla\tau_n\right]\nabla\tau_{n-1} \\ &\quad\leq \int_0^{\sigma(a)} \aleph(\tau_{n-2}, \tau_{n-1})\lambda(\tau_{n-1})\mathrm{g}_{n-1}(\Gamma_r)\nabla\tau_{n-1} \\ &\quad\leq \int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_{n-1})\lambda(\tau_{n-1})\mathrm{g}_{n-1}(\Gamma_r)\nabla\tau_{n-1} \\ &\quad\leq \frac{\mathfrak{N}_1\Gamma_r}{2}\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_{n-1})\prod_{i=1}^m \lambda_i(\tau_{n-1})\nabla\tau_{n-1} \\ &\quad\leq \frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^m \|\lambda_i\|_{L^{p_i}_\nabla} \leq \frac{\Gamma_r}{2} < \Gamma_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}&\int_0^{\sigma(a)} \aleph(t, \tau_1)\lambda(\tau_1)\mathrm{g}_1\left[\int_0^{\sigma(a)} \aleph(\tau_1, \tau_2)\lambda(\tau_2)\mathrm{g}_2\left[\int_0^{\sigma(a)} \aleph(\tau_2, \tau_3)\cdots\right.\right. \\ &\quad\left.\left.\times\, \mathrm{g}_{n-1}\left[\int_0^{\sigma(a)} \aleph(\tau_{n-1}, \tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\nabla\tau_n\right]\cdots\nabla\tau_3\right]\nabla\tau_2\right]\nabla\tau_1 \leq \frac{\Gamma_r}{2}.\end{aligned}$$

Also, we note that

$$\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\zeta_k,\tau_1)\lambda(\tau_1)\mathrm{g}_1\left[\int_0^{\sigma(a)}\aleph(\tau_1,\tau_2)\lambda(\tau_2)\cdots\right.$$
$$\left.\times\,\mathrm{g}_{n-1}\left[\int_0^{\sigma(a)}\aleph(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathrm{g}_n(\mathrm{x}_1(\tau_n))\nabla\tau_n\right]\cdots\nabla\tau_2\right]\nabla\tau_1$$
$$\leq\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\int_0^{\sigma(a)}\aleph(\tau_1,\tau_1)\lambda(\tau_1)\mathrm{g}_1(\Gamma_r)\nabla\tau_1$$
$$\leq\frac{1}{1-\sum_{k=1}^{n-2}c_k}\sum_{k=1}^{n-2}c_k\frac{\mathfrak{N}_1\Gamma_r}{2}\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^{p_i}_\nabla}\leq\frac{\Gamma_r}{2}.$$

Thus, $(\mathscr{L}\mathrm{x}_1)(t)\leq\frac{\Gamma_r}{2}+\frac{\Gamma_r}{2}=\Gamma_r$. Since $\Gamma_r=\|\mathrm{x}_1\|$ for $\mathrm{x}_1\in\mathtt{P}_{\eta_r}\cap\partial\mathtt{Q}_{1,r}$, we get

$$\|\mathscr{L}\mathrm{x}_1\|\leq\|\mathrm{x}_1\|. \tag{11}$$

Now define $\mathtt{Q}_{2,r}=\{\mathrm{x}_1\in\mathtt{X}:\|\mathrm{x}_1\|<\Lambda_r\}$. Let $\mathrm{x}_1\in\mathtt{P}_{\eta_r}\cap\partial\mathtt{Q}_{2,r}$ and let $\tau\in[\eta_r,\sigma(a)-\eta_r]_{\mathbb{T}}$. Then, the argument leading to (11) can be done to the present case. Hence, the theorem. □

Lastly, the case $\sum_{i=1}^{m}\frac{1}{p_i}>1$.

Theorem 3.5 *Suppose* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *hold, let* $\{\eta_r\}_{r=1}^{\infty}$ *be a sequence with* $t_{r+1}<\eta_r<t_r$. *Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Lambda_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1}<\frac{\eta_r}{\sigma(a)}\Lambda_r<\Lambda_r<\theta\Lambda_r<\Gamma_r\quad\text{and}\quad\frac{\eta_r}{\sigma(a)}<\frac{1}{2},\ r\in\mathbb{N}.$$

Assume that g_ℓ *satisfies* (J_2) *and*

(J_4) $\quad\mathrm{g}_\ell(\mathrm{x})\leq\frac{\mathfrak{N}_2\Gamma_r}{2}\ \forall\,t\in(0,\sigma(a)]_{\mathbb{T}},\ 0\leq\mathrm{x}\leq\Gamma_r$, where

$$\mathfrak{N}_2<\min\left\{\left[\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^1_\nabla}\right]^{-1},\left[\frac{\sum_{k=1}^{n-2}c_k}{1-\sum_{k=1}^{n-2}c_k}\|\aleph\|_{L^\infty_\nabla}\prod_{i=1}^{m}\|\lambda_i\|_{L^1_\nabla}\right]^{-1}\right\}.$$

Then the iterative boundary value problem (1)–(2) has infinitely many solutions $\{(\mathrm{x}_1^{[r]},\mathrm{x}_2^{[r]},\ldots,\mathrm{x}_n^{[r]})\}_{r=1}^{\infty}$ such that $\mathrm{x}_\ell^{[r]}(t)\geq0$ on $(0,\sigma(a)]_{\mathbb{T}}$, $\ell=1,2,\ldots,n$ and $r\in\mathbb{N}$.

Proof The proof is similar to the proof of Theorem 3.1. So, we omit the details here. □

4 Example

In this section, we provide two examples to check validity of our main results.

Example 4.1 Consider the following boundary value problem on $\mathbb{T}=[0,1]$.

$$\left.\begin{array}{l} x_\ell''(t) + \lambda(t) g_\ell(x_{\ell+1}(t)) = 0, t \in (0, \sigma(1)]_{\mathbb{T}}, \ \ell = 1, 2, 3, 4, \\ x_5(t) = x_1(t), t \in (0, \sigma(1)]_{\mathbb{T}}, \end{array}\right\} \tag{12}$$

$$x_\ell'(0) = 0, \ \ x_\ell(1) = \frac{1}{2} x_\ell\left(\frac{1}{3}\right) + \frac{1}{3} x_\ell\left(\frac{1}{4}\right), \tag{13}$$

where we take $n = 4, m = 2$, $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{3}$, $\zeta_1 = \frac{1}{3}$, $\zeta_2 = \frac{1}{4}$ and $\lambda(t) = \lambda_1(t)\lambda_2(t)$ in which

$$\lambda_1(t) = \frac{1}{|t - \frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \lambda_2(t) = \frac{1}{|t - \frac{3}{4}|^{\frac{1}{2}}}.$$

Then $\sum_{k=1}^{n-2} c_k = \frac{5}{6} < 1$ and $\delta_1 = \delta_2 = (4/3)^{1/2}$. For $\ell = 1, 2, 3, 4$, let

$$g_\ell(x) = \begin{cases} 0.05 \times 10^{-4}, x \in (10^{-4}, +\infty), \\ \frac{62\times 10^{-(4r+3)} - 0.05\times 10^{-4r}}{10^{-(4r+3)} - 10^{-4r}}(x - 10^{-4r}) + 0.05 \times 10^{-8r}, \\ \quad x \in \left[10^{-(4r+3)}, 10^{-4r}\right], \\ 62 \times 10^{-(4r+3)}, x \in \left(\frac{1}{5} \times 10^{-(4r+3)}, 10^{-(4r+3)}\right), \\ \frac{62\times 10^{-(4r+3)} - 0.05\times 10^{-8r}}{0.05\times 10^{-(4r+3)} - 10^{-(4r+4)}}(x - 10^{-(4r+4)}) + 0.05 \times 10^{-8r}, \\ \quad x \in \left(10^{-(4r+4)}, \frac{1}{5} \times 10^{-(4r+3)}\right], \\ 0, \ x = 0, \end{cases}$$

for all $r \in \mathbb{N}$. Let

$$t_r = \frac{31}{64} - \sum_{k=1}^{r} \frac{1}{4(k+1)^4} \ \text{ and } \ \eta_r = \frac{1}{2}(t_r + t_{r+1}), \ r \in \mathbb{N},$$

then

$$\eta_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}$$

and

$$t_{r+1} < \eta_r < t_r, \ \eta_r > \frac{1}{5}.$$

Therefore,

$$\frac{\eta_r}{a} = \frac{\eta_r}{1} > \frac{1}{5}, \ r \in \mathbb{N}.$$

It is clear that

$$t_1 = \frac{15}{32} < \frac{1}{2}, \ t_r - t_{r+1} = \frac{1}{4(r+2)^4}, \ r \in \mathbb{N}.$$

Since $\sum_{j=1}^{\infty}\frac{1}{j^4}=\frac{\pi^4}{90}$ and $\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6}$, it follows that

$$t^*=\lim_{r\to\infty}t_r=\frac{31}{64}-\sum_{k=1}^{\infty}\frac{1}{4(r+1)^4}=\frac{47}{64}-\frac{\pi^4}{360}=0.46.$$

Also, we have

$$\int_{\eta_1}^{\sigma(a)-\eta_1}\aleph(\tau,\tau)\Delta\tau=\int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}}(1-\tau)d\tau=0.03.$$

Thus, we get

$$\theta=\max\left\{\frac{1}{0.0163},\frac{1}{5\times 0.0163}\right\}=61.35.$$

Next, let $0<\mathfrak{a}<1$ be fixed. Then $\lambda_1,\lambda_2\in L^{1+\mathfrak{a}}[0,1]$. A simple calculations shows that

$$\int_0^{\sigma(1)}\lambda_1(t)\lambda_2(t)dt=\pi-\ln(7-4\sqrt{3}).$$

So, let $p_i=1$ for $i=1,2$. Then

$$\prod_{i=1}^{m}\|\lambda_i\|_{L_\nabla^{p_i}}=\pi-\ln(7-4\sqrt{3})\approx 5.78,$$

and also $\|\aleph\|_{L_\nabla^\infty}=1$. Therefore,

$$\mathfrak{N}_1<\left[\|\aleph\|_\infty\prod_{i=1}^{m}\|\lambda_i\|_{L_\nabla^{p_i}}\right]^{-1}\approx 0.173.$$

Taking $\mathfrak{N}_1=\frac{1}{10}$. In addition if we take

$$\Gamma_r=10^{-4r},\ \Lambda_r=10^{-(4r+3)},$$

then

$$\Gamma_{r+1}=10^{-(4r+4)}<\frac{1}{5}\times 10^{-(4r+3)}<\frac{\eta_r}{a}\Lambda_r$$
$$<\Lambda_r=10^{-(4r+3)}<\Gamma_r=10^{-4r},$$

$\theta\Lambda_r=61.35\times 10^{-(4r+3)}<\frac{1}{10}\times 10^{-4r}=\mathfrak{N}_1\Gamma_r$, $r\in\mathbb{N}$ and $g_\ell(\ell=1,2,3,4)$ satisfies the following growth conditions:

$$g_\ell(x)\le\mathfrak{N}_1\Gamma_r=\frac{1}{10}\times 10^{-4r},\quad x\in\left[0,10^{-4r}\right],$$
$$g_\ell(x)\ge\theta\Lambda_r=61.35\times 10^{-(4r+3)},\quad x\in\left[\frac{1}{5}\times 10^{-(4r+3)},10^{-(4r+3)}\right],$$

for $r \in \mathbb{N}$. Then all the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the iterative boundary value problem (1) has infinitely many solutions $\{(x_1^{[r]}, x_2^{[r]}, x_3^{[r]}, x_4^{[r]})\}_{r=1}^{\infty}$ such that $x_\ell^{[r]}(t) \geq 0$ on $[0, 1]$, $\ell = 1, 2, 3, 4$ and $r \in \mathbb{N}$.

Example 4.2 Let $\mathbb{T} = \{0\} \cup [1/2, 1] \cup \left\{ \frac{1}{2^{k+1}} : k \in \mathbb{N} \right\}$. Consider the boundary value problem

$$\left.\begin{array}{l} x_\ell^{\Delta\nabla}(t) + \lambda(t) g_\ell(x_{\ell+1}(t)) = 0,\ t \in (0, \sigma(1)]_{\mathbb{T}},\ \ell = 1, 2, 3, \\ x_4(t) = x_1(t),\ t \in (0, \sigma(1)]_{\mathbb{T}}, \end{array}\right\} \tag{14}$$

$$x'_\ell(0) = 0, \quad x_\ell(1) = \frac{1}{5} x_\ell\left(\frac{1}{4}\right), \tag{15}$$

where we take $n = 3$, $m = 2$, $c_1 = \frac{1}{5}$, $\zeta_1 = \frac{1}{4}$ and $\lambda(t) = \lambda_1(t)\lambda_2(t)$ in which

$$\lambda_1(t) = \frac{1}{|t - \frac{2}{5}|^{1/4}} \quad \text{and} \quad \lambda_2(t) = \frac{1}{|t - \frac{3}{4}|^{1/4}}.$$

Then $\sum_{k=1}^{n-2} c_k = \frac{1}{5} < 1$ and $\delta_1 = \delta_2 = (4/3)^{1/4}$. For $\ell = 1, 2, 3$, let

$$g_\ell(x) = \begin{cases} \frac{1}{5} \times 10^{-9}, x \in (10^{-9}, +\infty), \\ \frac{62 \times 10^{-(8r+3)} - \frac{1}{5} \times 10^{-(8r+1)}}{10^{-(8r+3)} - 10^{-(8r+1)}} (x - 10^{-(8r+1)}) + \frac{1}{5} \times 10^{-(8r+1)}, \\ x \in \left[10^{-(8r+3)}, 10^{-(8r+1)}\right], \\ 62 \times 10^{-(8r+3)}, x \in \left(\frac{1}{5} \times 10^{-(8r+3)}, 10^{-(8r+3)}\right), \\ \frac{62 \times 10^{-(8r+3)} - \frac{1}{5} \times 10^{-(8r+4)}}{\frac{1}{5} \times 10^{-(8r+3)} - 10^{-(8r+4)}} (x - 10^{-(8r+4)}) + \frac{1}{5} \times 10^{-(8r+4)}, \\ x \in \left(10^{-(8r+4)}, \frac{1}{5} \times 10^{-(8r+3)}\right], \\ 0, \ x = 0, \end{cases}$$

for all $r \in \mathbb{N}$.

Let t_r, η_r be the same as in example 4.1. Then $\eta_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}, t_{r+1} < \eta_r < t_r$, $\eta_r > \frac{1}{5}$ and $t_1 = \frac{15}{32} < \frac{1}{2}$, $t_r - t_{r+1} = \frac{1}{4(r+2)^4}$, $r \in \mathbb{N}$. Also, $t^* = \lim_{r\to\infty} t_r = \frac{31}{64} - \sum_{i=1}^{\infty} \frac{1}{4(i+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} = 0.46$. Also, we have

$$\int_{\eta_1}^{\sigma(a)-\eta_1} \aleph(\tau, \tau)\Delta\tau = \int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}} (1 - \tau) d\tau = 0.03.$$

Thus, we get

$$\theta = \max\left\{ \frac{1}{0.0161845}, \frac{1}{4 \times 0.0161845} \right\} = 61.79.$$

By Lemma 2.4, we obtain

$$\int_0^{\sigma(1)} \lambda_1(t)\lambda_2(t)dt = \int_{\frac{1}{2}}^{1} \lambda_1(t)\lambda_2(t)dt + \sum_{k=1}^{\infty} \left[\sigma\left(\frac{1}{2^k}\right) - \frac{1}{2^k}\right]\lambda_1\left(\frac{1}{2^k}\right)\lambda_2\left(\frac{1}{2^k}\right)$$
$$\approx 2.311909422$$

So, let $p_i = 1$ for $i = 1, 2$. Then

$$\prod_{i=1}^{m} \|\lambda_i\|_{L_\nabla^{p_i}} \approx 2.311909422,$$

and also $\|\aleph\|_{L_\nabla^\infty} = 1$. Therefore,

$$\mathfrak{N}_1 < \left[\|\aleph\|_\infty \prod_{i=1}^{m} \|\lambda_i\|_{L_\nabla^{p_i}}\right]^{-1} \approx 0.4325428974.$$

Taking $\mathfrak{N}_1 = \frac{1}{3}$. In addition, if we take

$$\Gamma_r = 10^{-8r} \text{ and } \Lambda_r = 10^{-(8r+3)},$$

then

$$\Gamma_{r+1} = 10^{-(8r+8)} < \frac{1}{5} \times 10^{-(8r+3)} < \frac{\eta_r}{a}\Lambda_r < \Lambda_r = 10^{-(8r+3)} < \Gamma_r = 10^{-8r},$$
$$\theta\Lambda_r = 61.79 \times 10^{-(8r+3)} < \frac{1}{3} \times 10^{-8r} = \mathfrak{N}_1\Gamma_r, \ r \in \mathbb{N}$$

and $g_\ell(\ell = 1, 2, 3)$ satisfies the following growth conditions:

$$g_\ell(x) \le \mathfrak{N}_1\Gamma_r = \frac{1}{3} \times 10^{-8r}, \ \ x \in \left[0, 10^{-8r}\right],$$
$$g_\ell(x) \ge \theta\Lambda_r = 61.79 \times 10^{-(8r+3)}, \ \ x \in \left[\frac{1}{5} \times 10^{-(8r+3)}, 10^{-(8r+3)}\right],$$

for $r \in \mathbb{N}$. Then all the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the iterative boundary value problem (1) has infinitely many solutions $\{(x_1^{[r]}, x_2^{[r]}, x_3^{[r]})\}_{r=1}^{\infty}$ such that $x_\ell^{[r]}(t) \ge 0$ on [0, 1], $\ell = 1, 2, 3$ and $r \in \mathbb{N}$.

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Infinitely many positive solutions for an iterative system of singular BVP on time scales

K. Rajendra Prasad[1]
Mahammad Khuddush[2]
K. V. Vidyasagar[3]

[1] *Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, 530003, India.*
rajendra92@rediffmail.com

[2] *Department of Mathematics, Dr. Lankapalli Bullayya College, Resapuvanipalem, Visakhapatnam, 530013, India.*
khuddush89@gmail.com

[3] *Department of Mathematics, S. V. L. N. S. Government Degree College, Bheemunipatnam, Bheemili, 531163, India.*
vidyavijaya08@gmail.com

ABSTRACT

In this paper, we consider an iterative system of singular two-point boundary value problems on time scales. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of infinitely many positive solutions. Finally, we provide an example to check the validity of our obtained results.

RESUMEN

En este artículo, consideramos un sistema iterativo de problemas de valor en la frontera singulares de dos puntos en escalas de tiempo. Aplicando la desigualdad de Hölder y el teorema de punto fijo cónico de Krasnoselskii en un espacio de Banach, derivamos condiciones suficientes para la existencia de una cantidad infinita de soluciones positivas. Finalmente, entregamos un ejemplo para verificar la validez de nuestros resultados.

Keywords and Phrases: Iterative system, time scales, singularity, cone, Krasnoselskii's fixed point theorem, positive solutions.

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1 Introduction

The theory of time scales was created to unify continuous and discrete analysis. Difference and differential equations can be studied simultaneously by studying dynamic equations on time scales. A time scale is any closed and nonempty subset of the real numbers. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles [1, 2] and the monographs of Bohner and Peterson [6, 7]. There is a great deal of research activity devoted to existence of solutions to the dynamic equations on time scales, see for example [8, 9, 13, 16–19] and references therein.

In [14], Liang and Zhang studied countably many positive solutions for nonlinear singular m–point boundary value problems on time scales,

$$\begin{aligned} \big(\varphi(\upsilon^{\Delta}(t))\big)^{\nabla} + a(t)f\big(\upsilon(t)\big) = 0,\ t \in [0, \mathfrak{T}]_{\mathbb{T}}, \\ \upsilon(0) = \sum_{i=1}^{m-2} a_i \upsilon(\xi_i),\ \upsilon^{\Delta}(\mathfrak{T}) = 0, \end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [12], Khuddush, Prasad and Vidyasagar considered second order n-point boundary value problem on time scales,

$$\begin{aligned} \upsilon_i^{\Delta\nabla}(t) + \uplambda(t)\mathrm{g}_\ell\big(\upsilon_{i+1}(t)\big) = 0,\ 1 \le i \le n,\ t \in (0, \upsigma(a)]_{\mathbb{T}}, \\ \upsilon_{n+1}(t) = \upsilon_1(t),\ t \in (0, \upsigma(a)]_{\mathbb{T}}, \\ \upsilon_i^{\Delta}(0) = 0,\ \upsilon_i(\upsigma(a)) = \sum_{k=1}^{n-2} c_k \upsilon_i(\upzeta_k),\ 1 \le i \le n, \end{aligned}$$

and established existence of positive solutions by applying Krasnoselskii's fixed point theorem.

Inspired by the aforementioned works, in this paper by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of infinitely many positive solutions for the iterative system of two-point boundary value problems with n–singularities on time scales,

$$\left.\begin{aligned} \upsilon_\ell^{\Delta\Delta}(t) + \uplambda(t)\mathrm{g}_\ell\big(\upsilon_{\ell+1}(t)\big) = 0,\ 1 \le \ell \le m,\ t \in (0, \mathfrak{T})_{\mathbb{T}}, \\ \upsilon_{m+1}(t) = \upsilon_1(t),\ t \in (0, \mathfrak{T})_{\mathbb{T}}, \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \upsilon_\ell(0) = \upsilon_\ell^{\Delta}(0),\ 1 \le \ell \le m, \\ \upsilon_\ell(\mathfrak{T}) = -\upsilon_\ell^{\Delta}(\mathfrak{T}),\ 1 \le \ell \le m, \end{aligned}\right\} \tag{1.2}$$

where $m \in \mathbb{N}$, $\lambda(t) = \prod_{i=1}^{k} \lambda_i(t)$ and each $\lambda_i(t) \in L_{\Delta}^{p_i}([0, \mathfrak{T}]_{\mathbb{T}})$ $(p_i \geq 1)$ has n–singularities in the interval $(0, \mathfrak{T})_{\mathbb{T}}$.

We assume the following conditions are true throughout the paper:

(H_1) $\mathsf{g}_\ell : [0, +\infty) \to [0, +\infty)$ is continuous.

(H_2) $\lim_{t \to t_i} \lambda_i(t) = \infty$, where $0 < t_n < t_{n-1} < \cdots < t_1 < \mathfrak{T}$.

2 Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions.

Definition 2.1 ([6]). *A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to [0, +\infty)$ are defined by*

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\},$$

$$\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\},$$

and

$$\mu(t) = \sigma(t) - t,$$

respectively.

- *The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.*

- *If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$.*

- *If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.*

- *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.*

- *A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.*

- *By an interval time scale, we mean the intersection of a real interval with a given time scale, i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. Other intervals can be defined similarly.*

Definition 2.2 ([5,11]). *Let μ_Δ and μ_∇ be the Lebesgue $\Delta-$measure and the Lebesgue $\nabla-$measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.*

(i) If there exists $\Gamma_1 \subset A$ with $\mu_\Delta(\Gamma_1) = 0$ such that P holds on $A\backslash\Gamma_1$, then P is said to hold Δ–a.e. on A.

(ii) If there exists $\Gamma_2 \subset A$ with $\mu_\nabla(\Gamma_2) = 0$ such that P holds on $A\backslash\Gamma_2$, then P is said to hold ∇–a.e. on A.

Definition 2.3 ([4,5]). *Let $E \subset \mathbb{T}$ be a $\Delta-$measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be a $\Delta-$measurable function. We say that f belongs to $L^p_\Delta(E)$ provided that either*

$$\int_E |f|^p(s)\Delta s < \infty \quad \textit{if} \quad p \in [1, +\infty),$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M, \quad \Delta - a.e. \text{ on } E \quad \textit{if} \quad p = +\infty.$$

Lemma 2.4 ([20]). *Let $E \subset \mathbb{T}$ be a $\Delta-$measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is $\Delta-$integrable on E, then*

$$\int_E f(s)\Delta s = \int_E f(s)ds + \sum_{i \in I_E} \big(\sigma(t_i) - t_i\big) f(t_i) + r(f, E),$$

where

$$r(f, E) = \begin{cases} \mu_{\mathtt{N}}(E) f(M), & \textit{if } \mathtt{N} \in \mathbb{T}, \\ \\ 0, & \textit{if } \mathtt{N} \notin \mathbb{T}, \end{cases}$$

$I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}$, $I \subset \mathbb{N}$, is the set of all right-scattered points of $\mathbb{T}$.

Lemma 2.5. *For any $\mathtt{y}(t) \in \mathcal{C}_{rd}([0, \mathfrak{T}]_{\mathbb{T}})$, the boundary value problem,*

$$\upsilon_1^{\Delta\Delta}(t) + \mathtt{y}(t) = 0, \ t \in (0, \mathfrak{T})_{\mathbb{T}}, \tag{2.1}$$

$$\upsilon_1(0) = \upsilon_1^{\Delta}(0), \ \upsilon_1(\mathfrak{T}) = -\upsilon_1^{\Delta}(\mathfrak{T}), \tag{2.2}$$

has a unique solution

$$\upsilon_1(t) = \int_0^{\mathfrak{T}} \aleph(t, \tau)\mathtt{y}(\tau)\Delta\tau, \tag{2.3}$$

where

$$\aleph(t, \tau) = \frac{1}{2 + \mathfrak{T}} \begin{cases} (\mathfrak{T} - t + 1)(\sigma(\tau) + 1), & \textit{if} \ \ \sigma(\tau) < t, \\ (\mathfrak{T} - \sigma(\tau) + 1)(t + 1), & \textit{if} \ \ t < \tau. \end{cases} \tag{2.4}$$

Proof. Suppose υ_1 is a solution of (2.1), then

$$\begin{aligned}\upsilon_1(t) &= -\int_0^t \int_0^\tau \mathrm{y}(\tau_1)\Delta\tau_1\Delta\tau + A_1 t + A_2\\ &= -\int_0^t (t-\sigma(\tau))\mathrm{y}(\tau)\Delta\tau + A_1 t + A_2,\end{aligned}$$

where $A_1 = \upsilon_1^\Delta(0)$ and $A_2 = \upsilon_1(0)$. By the conditions (2.2), we get

$$A_1 = A_2 = \frac{1}{2+\mathfrak{T}}\int_0^{\mathfrak{T}} (\mathfrak{T}-\sigma(\tau)+1)\mathrm{y}(\tau)\Delta\tau.$$

So, we have

$$\begin{aligned}\upsilon_1(t) &= \int_0^t (t-\sigma(\tau))\mathrm{y}(\tau)\Delta\tau + \frac{1}{2+\mathfrak{T}}\int_0^{\mathfrak{T}} (\mathfrak{T}-\sigma(\tau)+1)(1+t)\mathrm{y}(\tau)\Delta\tau\\ &= \int_0^{\mathfrak{T}} \aleph(t,\tau)\mathrm{y}(\tau)\Delta\tau.\end{aligned}$$

This completes the proof. □

Lemma 2.6. *Suppose* (H_1)*–*(H_2) *hold. For* $\varepsilon \in (0, \frac{\mathfrak{T}}{2})_{\mathbb{T}}$*, let* $\mathcal{G}(\varepsilon) = \dfrac{\varepsilon+1}{\mathfrak{T}+1} < 1$. *Then* $\aleph(t,\tau)$ *has the following properties:*

(i) $0 \le \aleph(t,\tau) \le \aleph(\tau,\tau)$ *for all* $t,\tau \in [0,1]_{\mathbb{T}}$,

(ii) $\mathcal{G}(\varepsilon)\aleph(\tau,\tau) \le \aleph(t,\tau)$ *for all* $t \in [\varepsilon, \mathfrak{T}-\varepsilon]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$.

Proof. *(i)* is evident. To prove *(ii)*, let $t \in [\varepsilon, \mathfrak{T}-\varepsilon]_{\mathbb{T}}$ and $t \le \tau$. Then

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)} = \frac{t+1}{\tau+1} \ge \frac{\varepsilon+1}{\mathfrak{T}+1} = \mathcal{G}(\varepsilon).$$

For $\tau \le t$,

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)} = \frac{\mathfrak{T}-t+1}{\mathfrak{T}-\tau+1} \ge \frac{\varepsilon+1}{\mathfrak{T}+1} = \mathcal{G}(\varepsilon).$$

This completes the proof. □

Notice that an $m-$tuple $(\upsilon_1(t), \upsilon_2(t), \upsilon_3(t), \ldots, \upsilon_m(t))$ is a solution of the iterative boundary value problem (1.1)–(1.2) if and only if

$$\begin{aligned}\upsilon_\ell(t) &= \int_0^1 \aleph(t,\tau)\lambda(\tau)\mathrm{g}_\ell(\upsilon_{\ell+1}(\tau))\Delta\tau,\ t \in (0,\mathfrak{T})_{\mathbb{T}},\ 1 \le \ell \le m,\\ \upsilon_{m+1}(t) &= \upsilon_1(t),\ t \in (0,\mathfrak{T})_{\mathbb{T}},\end{aligned}$$

i.e.,

$$\begin{aligned}\upsilon_1(t) = \int_0^1 &\aleph(t,\tau_1)\lambda(\tau_1)\mathrm{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathrm{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ &\times \mathrm{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

Let $\mathtt{B}$ be the Banach space $C_{rd}((0,\mathfrak{T})_{\mathbb{T}},\mathbb{R})$ with the norm $\|\upsilon\| = \max_{t\in(0,\mathfrak{T})_{\mathbb{T}}} |\upsilon(t)|$. For $\varepsilon \in \left(0, \frac{\mathfrak{T}}{2}\right)_{\mathbb{T}}$, we define the cone $\mathtt{K}_\varepsilon \subset \mathtt{B}$ as

$$\mathtt{K}_\varepsilon = \left\{\upsilon \in \mathtt{B} : \upsilon(t) \text{ is nonnegative and } \min_{t\in[\varepsilon,\,\mathfrak{T}-\varepsilon]_{\mathbb{T}}} \upsilon(t) \geq \mathcal{G}(\varepsilon)\|\upsilon(t)\|\right\}.$$

For any $\upsilon_1 \in \mathtt{K}_\varepsilon$, define an operator $\Omega : \mathtt{K}_\varepsilon \to \mathtt{B}$ by

$$\begin{aligned}(\Omega\upsilon_1)(t) = \int_0^1 \aleph(t,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots \\ \times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

Lemma 2.7. *Assume that (H_1)–(H_2) hold. Then for each $\varepsilon \in \left(0, \frac{\mathfrak{T}}{2}\right)_{\mathbb{T}}$, $\Omega(\mathtt{K}_\varepsilon) \subset \mathtt{K}_\varepsilon$ and $\Omega : \mathtt{K}_\varepsilon \to \mathtt{K}_\varepsilon$ are completely continuous.*

Proof. From Lemma 2.6, $\aleph(t,\tau) \geq 0$ for all $t,\tau \in (0,\mathfrak{T})_{\mathbb{T}}$. So, $(\Omega\upsilon_1)(t) \geq 0$. Also, for $\upsilon_1 \in \mathtt{K}_\varepsilon$, we have

$$\begin{aligned}\|\Omega\upsilon_1\| &= \max_{t\in(0,\mathfrak{T})_{\mathbb{T}}} \int_0^1 \aleph(t,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots \\ &\quad \times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1 \\ &\leq \int_0^1 \aleph(\tau_1,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots \\ &\quad \times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

Again from Lemma 2.6, we get

$$\begin{aligned}\min_{t\in[\varepsilon,\mathfrak{T}-\varepsilon]_{\mathbb{T}}} \big\{(\Omega\upsilon_1)(t)\big\} \geq \mathcal{G}(\varepsilon)\int_0^1 \aleph(\tau_1,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots \\ \times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

It follows from the above two inequalities that

$$\min_{t\in[\varepsilon,\mathfrak{T}-\varepsilon]_{\mathbb{T}}} \big\{(\Omega\upsilon_1)(t)\big\} \geq \mathcal{G}(\varepsilon)\|\Omega\upsilon_1\|.$$

So, $\Omega\upsilon_1 \in \mathtt{K}_\varepsilon$ and thus $\Omega(\mathtt{K}_\varepsilon) \subset \mathtt{K}_\varepsilon$. Next, by standard methods and the Arzela-Ascoli theorem, it can be proved easily that the operator Ω is completely continuous. The proof is complete. □

3 Infinitely many positive solutions

For the existence of infinitely many positive solutions for iterative system of boundary value problem (1.1)–(1.2), we apply following theorems.

Theorem 3.1 ([10]). *Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and let $\mathtt{M}_1, \mathtt{M}_2$ be open sets with $0 \in \mathtt{M}_1, \overline{\mathtt{M}}_1 \subset \mathtt{M}_2$. Let $\mathcal{A} : \mathcal{E} \cap (\overline{\mathtt{M}}_2 \backslash \mathtt{M}_1) \to \mathcal{E}$ be a completely continuous operator such that* *(a)* $\|\mathcal{A}v\| \leq \|v\|$, $v \in \mathcal{E} \cap \partial\mathtt{M}_1$, *and* $\|\mathcal{A}v\| \geq \|v\|$, $v \in \mathcal{E} \cap \partial\mathtt{M}_2$, *or*

(b) $\|\mathcal{A}v\| \geq \|v\|$, $v \in \mathcal{E} \cap \partial\mathtt{M}_1$, *and* $\|\mathcal{A}v\| \leq \|v\|$, $v \in \mathcal{E} \cap \partial\mathtt{M}_2$.

Then $\mathcal{A}$ *has a fixed point in* $\mathcal{E} \cap (\overline{\mathtt{M}}_2 \backslash \mathtt{M}_1)$.

Theorem 3.2 ([7,15]). *Let* $f \in L^p_\nabla(J)$ *with* $p > 1$, $g \in L^q_\Delta(J)$ *with* $q > 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $fg \in L^1_\Delta(J)$ *and* $\|fg\|_{L^1_\Delta} \leq \|f\|_{L^p_\Delta}\|g\|_{L^q_\Delta}$, *where*

$$\|f\|_{L^p_\Delta} := \begin{cases} \left[\int_J |f|^p(s)\Delta s\right]^{\frac{1}{p}}, & p \in \mathbb{R}, \\ \inf\left\{M \in \mathbb{R} \,/\, |f| \leq M \ \Delta - a.e. \ on \ J\right\}, & p = \infty, \end{cases}$$

and $J = [a, b)_{\mathbb{T}}$.

Theorem 3.3 (Hölder's inequality [3,4,15]). *Let* $f \in L^{p_i}_\Delta(J)$ *with* $p_i > 1$, *for* $i = 1, 2, \ldots, n$ *and* $\sum_{i=1}^n \frac{1}{p_i} = 1$. *Then* $\prod_{i=1}^k \mathtt{g}_i \in L^1_\Delta(J)$ *and* $\left\|\prod_{i=1}^k \mathtt{g}_i\right\|_1 \leq \prod_{i=1}^k \|\mathtt{g}_i\|_{p_i}$. *Further, if* $f \in L^1_\Delta(J)$ *and* $g \in L^\infty_\Delta(J)$, *then* $fg \in L^1_\Delta(J)$ *and* $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$.

We need the following condition in the sequel:

(H_3) There exists $\delta_i > 0$ such that $\lambda_i(t) > \delta_i$ $(i = 1, 2, \ldots, n)$ for $t \in [0, \mathfrak{T}]_{\mathbb{T}}$.

Consider the following three possible cases for $\lambda_i \in L^{p_i}_\Delta(0, \mathfrak{T})_{\mathbb{T}}$:

$$\sum_{i=1}^n \frac{1}{p_i} < 1, \quad \sum_{i=1}^n \frac{1}{p_i} = 1, \quad \sum_{i=1}^n \frac{1}{p_i} > 1.$$

Firstly, we seek infinitely many positive solutions for the case $\displaystyle\sum_{i=1}^n \frac{1}{p_i} < 1$.

Theorem 3.4. *Suppose* (H_1)–(H_3) *hold, let* $\{\varepsilon_r\}_{r=1}^\infty$ *be such that* $0 < \varepsilon_1 < \mathfrak{T}/2$, $\varepsilon \downarrow t^*$ *and* $0 < t^* < t_n$. *Let* $\{\Gamma_r\}_{r=1}^\infty$ *and* $\{\Lambda_r\}_{r=1}^\infty$ *be such that*

$$\Gamma_{r+1} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r < \theta\Lambda_r < \Gamma_r, \ r \in \mathbb{N},$$

where

$$\theta = \max\left\{\left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^k \delta_i \int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau, \tau)\Delta\tau\right]^{-1}, \ 1\right\}.$$

Assume that $\mathtt{g}_\ell$ *satisfies*

(C_1) $\mathtt{g}_\ell(\upsilon) \leq \mathfrak{N}_1\Gamma_r \ \forall \ t \in (0, \mathfrak{T})_{\mathbb{T}}$, $0 \leq \upsilon \leq \Gamma_r$, *where*

$$\mathfrak{N}_1 < \left[\|\aleph\|_{L^q_\Delta}\prod_{i=1}^k \|\lambda_i\|_{L^{p_i}_\Delta}\right]^{-1},$$

(C_2) $\mathtt{g}_\ell(\upsilon) \geq \theta\Lambda_r \ \forall \ t \in [\varepsilon_r, \mathfrak{T} - \varepsilon_r]_{\mathbb{T}}$, $\mathcal{G}(\varepsilon_r)\Lambda_r \leq \upsilon \leq \Lambda_r$.

Then the iterative boundary value problem (1.1)–(1.2) *has infinitely many solutions* $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]}, \dots, \upsilon_m^{[r]})\}_{r=1}^{\infty}$ *such that* $\upsilon_\ell^{[r]}(t) \geq 0$ *on* $(0, \mathfrak{T})_{\mathbb{T}}$, $\ell = 1, 2, \dots, m$ *and* $r \in \mathbb{N}$.

Proof. Let

$$\mathtt{M}_{1,r} = \{\upsilon \in \mathtt{B} : \|\upsilon\| < \Gamma_r\}, \quad \mathtt{M}_{2,r} = \{\upsilon \in \mathtt{B} : \|\upsilon\| < \Lambda_r\},$$

be open subsets of $\mathtt{B}$. Let $\{\varepsilon_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^* < t_{r+1} < \varepsilon_r < t_r < \frac{\mathfrak{T}}{2},$$

for all $r \in \mathbb{N}$. For each $r \in \mathbb{N}$, we define the cone $\mathtt{K}_{\varepsilon_r}$ by

$$\mathtt{K}_{\varepsilon_r} = \Big\{\upsilon \in \mathtt{B} : \upsilon(t) \geq 0, \min_{t \in [\varepsilon_r, \mathfrak{T} - \varepsilon_r]_{\mathbb{T}}} \upsilon(t) \geq \mathcal{G}(\varepsilon_r)\|\upsilon(t)\|\Big\}.$$

Let $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial \mathtt{M}_{1,r}$. Then, $\upsilon_1(\tau) \leq \Gamma_r = \|\upsilon_1\|$ for all $\tau \in (0, \mathfrak{T})_{\mathbb{T}}$. By (C_1) and for $\tau_{m-1} \in (0, \mathfrak{T})_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m &\leq \int_0^{\mathfrak{T}} \aleph(\tau_m, \tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m \\ &\leq \mathfrak{N}_1\Gamma_r \int_0^{\mathfrak{T}} \aleph(\tau_m, \tau_m) \prod_{i=1}^{k} \lambda_i(\tau_m)\Delta\tau_m.\end{aligned}$$

There exists a $q > 1$ such that $\dfrac{1}{q} + \displaystyle\sum_{i=1}^{n} \frac{1}{p_i} = 1$. So,

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m &\leq \mathfrak{N}_1\Gamma_r \big\|\aleph\big\|_{L^q_\Delta} \left\|\prod_{i=1}^{k}\lambda_i\right\|_{L^{p_i}_\Delta} \\ &\leq \mathfrak{N}_1\Gamma_r\|\aleph\|_{L^q_\Delta} \prod_{i=1}^{k} \|\lambda_i\|_{L^{p_i}_\Delta} \leq \Gamma_r.\end{aligned}$$

It follows in similar manner (for $\tau_{m-2} \in (0, \mathfrak{T})_{\mathbb{T}}$), that

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-2}, \tau_{m-1})\lambda(\tau_{m-1})\mathrm{g}_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\Delta\tau_{m-1} \\ \leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-2}, \tau_{m-1})\lambda(\tau_{m-1})\mathrm{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\ \leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_{m-1})\lambda(\tau_{m-1})\mathrm{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\ \leq \mathfrak{N}_1\Gamma_r \int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_{m-1}) \prod_{i=1}^{k} \lambda_i(\tau_{m-1})\Delta\tau_{m-1} \\ \leq \mathfrak{N}_1\Gamma_r\|\aleph\|_{L^q_\Delta} \prod_{i=1}^{k} \|\lambda_i\|_{L^{p_i}_\Delta} \leq \Gamma_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\Omega\upsilon_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t,\tau_1)\lambda(\tau_1)g_1\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\lambda(\tau_2)g_2\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\cdots \\ &\quad \times g_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)g_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1 \\ &\le \Gamma_r.\end{aligned}$$

Since $\Gamma_r = \|\upsilon_1\|$ for $\upsilon_1 \in K_{\varepsilon_r} \cap \partial M_{1,r}$, we get

$$\|\Omega\upsilon_1\| \le \|\upsilon_1\|. \tag{3.1}$$

Let $t \in [\varepsilon_r, \mathfrak{T}-\varepsilon_r]_{\mathbb{T}}$. Then,

$$\Lambda_r = \|\upsilon_1\| \ge \upsilon_1(t) \ge \min_{t\in[\varepsilon_r,\mathfrak{T}-\varepsilon_r]_{\mathbb{T}}} \upsilon_1(t) \ge \mathcal{G}(\varepsilon_r)\,\|\upsilon_1\| \ge \mathcal{G}(\varepsilon_r)\Lambda_r.$$

By (C_2) and for $\tau_{m-1} \in [\varepsilon_r, \mathfrak{T}-\varepsilon_r]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)g_m(\upsilon_1(\tau_m))\Delta\tau_m &\ge \int_{\varepsilon_r}^{\mathfrak{T}-\varepsilon_r} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)g_m(\upsilon_1(\tau_m))\Delta\tau_m \\ &\ge \mathcal{G}(\varepsilon_r)\theta\Lambda_r \int_{\varepsilon_r}^{\mathfrak{T}-\varepsilon_r} \aleph(\tau_m,\tau_m)\lambda(\tau_m)\Delta\tau_m \\ &\ge \mathcal{G}(\varepsilon_r)\theta\Lambda_r \int_{\varepsilon_r}^{\mathfrak{T}-\varepsilon_r} \aleph(\tau_m,\tau_m)\prod_{i=1}^{k}\lambda_i(\tau_m)\Delta\tau_m \\ &\ge \mathcal{G}(\varepsilon_1)\theta\Lambda_r \prod_{i=1}^{k}\delta_i \int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau_m,\tau_m)\Delta\tau_m \\ &\ge \Lambda_r.\end{aligned}$$

Continuing with the bootstrapping argument, we get

$$\begin{aligned}(\Omega\upsilon_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t,\tau_1)\lambda(\tau_1)g_1\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\lambda(\tau_2)g_2\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\cdots \\ &\quad \times g_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)g_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1 \\ &\ge \Lambda_r.\end{aligned}$$

Thus, if $\upsilon_1 \in K_{\varepsilon_r} \cap \partial K_{2,r}$, then

$$\|\Omega\upsilon_1\| \ge \|\upsilon_1\|. \tag{3.2}$$

It is evident that $0 \in M_{2,k} \subset \overline{M}_{2,k} \subset M_{1,k}$. From (3.1)–(3.2), it follows from Theorem 3.1 that the operator Ω has a fixed point $\upsilon_1^{[r]} \in K_{\varepsilon_r} \cap \left(\overline{M}_{1,r}\backslash M_{2,r}\right)$ such that $\upsilon_1^{[r]}(t) \ge 0$ on $(0,\mathfrak{T})_{\mathbb{T}}$, and $r \in \mathbb{N}$. Next setting $\upsilon_{m+1} = \upsilon_1$, we obtain infinitely many positive solutions $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]}, \ldots, \upsilon_m^{[r]})\}_{r=1}^{\infty}$ of (1.1)–(1.2) given iteratively by

$$\upsilon_\ell(t) = \int_0^{\mathfrak{T}} \aleph(t,\tau)\lambda(\tau)g_\ell(\upsilon_{\ell+1}(\tau))\Delta\tau,\ t \in (0,\mathfrak{T})_{\mathbb{T}},\ \ell = m, m-1, \ldots, 1.$$

The proof is completed. □

For $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, we have the following theorem.

Theorem 3.5. *Suppose* (H_1)–(H_3) *hold, let* $\{\varepsilon_r\}_{r=1}^{\infty}$ *be such that* $0 < \varepsilon_1 < \mathfrak{T}/2$, $\varepsilon \downarrow t^*$ *and* $0 < t^* < t_n$. *Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Lambda_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r < \theta\Lambda_r < \Gamma_r, \ r \in \mathbb{N},$$

where

$$\theta = \max\left\{ \left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^{k}\delta_i \int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau,\tau)\Delta\tau\right]^{-1}, \ 1\right\}.$$

Assume that $\mathtt{g}_\ell$ *satisfies* (C_2) *and*

(C_3) $\mathtt{g}_j(\upsilon) \le \mathfrak{N}_2\Gamma_r \ \forall \ t \in (0,\mathfrak{T})_{\mathbb{T}}$, $0 \le \upsilon \le \Gamma_r$, *where*

$$\mathfrak{N}_2 < \min\left\{\left[\|\aleph\|_{L^{\infty}_{\Delta}} \prod_{i=1}^{k} \|\lambda_i\|_{L^{p_i}_{\Delta}}\right]^{-1}, \theta\right\}.$$

Then the iterative boundary value problem (1.1)–(1.2) *has infinitely many solutions* $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]}, \dots, \upsilon_m^{[r]})\}_{r=1}^{\infty}$ *such that* $\upsilon_\ell^{[r]}(t) \ge 0$ *on* $(0,\mathfrak{T})_{\mathbb{T}}$, $\ell = 1, 2, \dots, m$ *and* $r \in \mathbb{N}$.

Proof. For a fixed r, let $\mathtt{M}_{1,r}$ be as in the proof of Theorem 3.4 and let $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{2,r}$. Again

$$\upsilon_1(\tau) \le \Gamma_r = \|\upsilon_1\|,$$

for all $\tau \in (0,\mathfrak{T})_{\mathbb{T}}$. By (C_3) and for $\tau_{\ell-1} \in (0,\mathfrak{T})_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m &\le \int_0^{\mathfrak{T}} \aleph(\tau_m,\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m \\ &\le \mathfrak{N}_1\Gamma_r \int_0^{\mathfrak{T}} \aleph(\tau_m,\tau_m)\prod_{i=1}^{k}\lambda_i(\tau_m)\Delta\tau_m \\ &\le \mathfrak{N}_1\Gamma_r \big\|\aleph\big\|_{L^{\infty}_{\Delta}} \left\|\prod_{i=1}^{k}\lambda_i\right\|_{L^{p_i}_{\Delta}} \\ &\le \mathfrak{N}_1\Gamma_r \|\aleph\|_{L^{\infty}_{\Delta}} \prod_{i=1}^{k}\|\lambda_i\|_{L^{p_i}_{\Delta}} \le \Gamma_r.\end{aligned}$$

It follows in similar manner (for $\tau_{m-2} \in [0,1]_{\mathbb{T}}$), that

$$\begin{aligned}
\int_0^{\mathfrak{T}} \aleph(\tau_{m-2},\tau_{m-1})\lambda(\tau_{m-1})g_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)g_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\Delta\tau_{m-1} \\
&\le \int_0^{\mathfrak{T}} \aleph(\tau_{m-2},\tau_{m-1})\lambda(\tau_{m-1})g_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\
&\le \int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_{m-1})\lambda(\tau_{m-1})g_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\
&\le \mathfrak{N}_1\Gamma_r \int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_{m-1})\prod_{i=1}^{k}\lambda_i(\tau_{m-1})\Delta\tau_{m-1} \\
&\le \mathfrak{N}_1\Gamma_r\|\aleph\|_{L_\Delta^\infty}\prod_{i=1}^{k}\|\lambda_i\|_{L_\Delta^{p_i}} \le \Gamma_r.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\Omega\upsilon_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t,\tau_1)\lambda(\tau_1)g_1\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\lambda(\tau_2)g_2\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\cdots \\
&\quad \times g_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)g_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1 \\
&\le \Gamma_r.
\end{aligned}$$

Since $\Gamma_r = \|\upsilon_1\|$ for $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{1,r}$, we get

$$\|\Omega\upsilon_1\| \le \|\upsilon_1\|. \tag{3.3}$$

Now define $\mathtt{M}_{2,r} = \{\upsilon_1 \in \mathtt{B} : \|\upsilon_1\| < \Lambda_r\}$. Let $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{2,r}$ and let $\tau \in [\varepsilon_r, \mathfrak{T}-\varepsilon_r]_{\mathbb{T}}$. Then, the argument leading to (3.2) can be done to the present case. Hence, the theorem. □

Lastly, the case $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i} > 1$.

Theorem 3.6. *Suppose (H_1)–(H_3) hold, let $\{\varepsilon_r\}_{r=1}^{\infty}$ be such that $0 < \varepsilon_1 < \mathfrak{T}/2$, $\varepsilon \downarrow t^*$ and $0 < t^* < t_n$. Let $\{\Gamma_r\}_{r=1}^{\infty}$ and $\{\Lambda_r\}_{r=1}^{\infty}$ be such that*

$$\Gamma_{r+1} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r < \theta\Lambda_r < \Gamma_r,\ r \in \mathbb{N},$$

where

$$\theta = \max\left\{\left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^{k}\delta_i\int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1}\aleph(\tau,\tau)\Delta\tau\right]^{-1},\ 1\right\}.$$

Assume that g_ℓ satisfies (C_2) and

(C_4) $g_j(\upsilon) \le \mathfrak{N}_3\Gamma_r\ \forall\ t \in (0,\mathfrak{T})_{\mathbb{T}},\ 0 \le \upsilon \le \Gamma_r,$ *where* $\mathfrak{N}_3 < \min\left\{\left[\|\aleph\|_{L_\Delta^\infty}\prod_{i=1}^{k}\|\lambda_i\|_{L_\Delta^1}\right]^{-1}, \theta\right\}$.

Then the iterative boundary value problem (1.1)–(1.2) has infinitely many solutions $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]}, \ldots, \upsilon_m^{[r]})\}_{r=1}^{\infty}$ such that $\upsilon_\ell^{[r]}(t) \ge 0$ on $(0,\mathfrak{T})_{\mathbb{T}}$, $\ell = 1,2,\ldots,m$ and $r \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.4. So, we omit the details here. □

4 Example

In this section, we present an example to check validity of our main results.

Example 4.1. Consider the following boundary value problem on $\mathbb{T} = \mathbb{R}$.

$$\left.\begin{aligned} \upsilon_\ell''(t) + \lambda(t)g_\ell(\upsilon_{\ell+1}(t)) = 0,\ \ell = 1, 2, \\ \upsilon_3(t) = \upsilon_1(t), \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \upsilon_\ell(0) = \upsilon_\ell'(0), \\ \upsilon_\ell(1) = -\upsilon_\ell'(1), \end{aligned}\right\} \tag{4.2}$$

where

$$\lambda(t) = \lambda_1(t)\lambda_2(t)$$

in which

$$\lambda_1(t) = \frac{1}{|t - \frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \lambda_2(t) = \frac{1}{|t - \frac{3}{4}|^{\frac{1}{2}}},$$

$$g_1(\upsilon) = g_2(\upsilon) = \begin{cases} \frac{1}{5} \times 10^{-4}, & \upsilon \in (10^{-4}, +\infty), \\ \frac{25\times10^{-(4r+3)} - \frac{1}{5}\times10^{-4r}}{10^{-(4r+3)} - 10^{-4r}}(\upsilon - 10^{-4r}) + & \\ \frac{1}{5} \times 10^{-8r}, & \upsilon \in \left[10^{-(4r+3)}, 10^{-4r}\right], \\ 25 \times 10^{-(4r+3)}, & \upsilon \in \left(\frac{1}{5} \times 10^{-(4r+3)}, 10^{-(4r+3)}\right), \\ \frac{25\times10^{-(4r+3)} - \frac{1}{5}\times10^{-8r}}{\frac{1}{5}\times10^{-(4r+3)} - 10^{-(4r+4)}}(\upsilon - 10^{-(4r+4)}) + & \\ \frac{1}{5} \times 10^{-8r}, & \upsilon \in \left(10^{-(4r+4)}, \frac{1}{5} \times 10^{-(4r+3)}\right], \\ 0, & \upsilon = 0. \end{cases}$$

Let

$$t_r = \frac{31}{64} - \sum_{k=1}^{r} \frac{1}{4(k+1)^4}, \quad \varepsilon_r = \frac{1}{2}(t_r + t_{r+1}), \quad r = 1, 2, 3, \dots,$$

then

$$\varepsilon_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32},$$

and

$$t_{r+1} < \varepsilon_r < t_r, \quad \varepsilon_r > \frac{1}{5}.$$

Therefore,

$$\mathcal{G}(\varepsilon_r) = \frac{\varepsilon_r + 1}{\mathfrak{T} + 1} = \frac{\varepsilon_r + 1}{2} > \frac{1}{5}, \quad r = 1, 2, 3, \dots$$

It is clear that

$$t_1 = \frac{15}{32} < \frac{1}{2}, \quad t_r - t_{r+1} = \frac{1}{4(r+2)^4}, \quad r = 1, 2, 3, \dots$$

Since $\sum_{x=1}^{\infty} \frac{1}{x^4} = \frac{\pi^4}{90}$ and $\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{r\to\infty} t_r = \frac{31}{64} - \sum_{k=1}^{\infty} \frac{1}{4(r+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} = 0.4637941914,$$

$$\lambda_1, \lambda_2 \in L^p[0,1] \;\text{ for all }\; 0 < p < 2, \;\text{ and }\; \delta_1 = \delta_2 = (4/3)^{1/4},$$

$$\mathcal{G}(\varepsilon_1) = 0.7336033951.$$

$$\int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau,\tau)\Delta\tau = \int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}} \frac{(2-\tau)(1+\tau)}{3} d\tau = 0.04918197801.$$

Thus, we get

$$\begin{aligned}\theta &= \max\left\{ \left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^{k}\delta_i \int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau,\tau)\nabla\tau\right]^{-1}, \; 1\right\} \\ &= \max\left\{\frac{1}{0.04166167167}, 1\right\} \\ &= 24.00287746.\end{aligned}$$

Next, let $0 < \mathfrak{a} < 1$ be fixed. Then $\lambda_1, \lambda_2 \in L^{1+\mathfrak{a}}[0,1]$ and $\frac{2}{1+\mathfrak{a}} > 1$ for $0 < \mathfrak{a} < 1$. It follows that

$$\prod_{i=1}^{k} \|\lambda_i\|_{L^{p_i}_{\Delta}} \approx \pi - \ln(7 - 4\sqrt{3}),$$

and also $\|\aleph\|_\infty = \frac{2}{3}$. So, for $0 < \mathfrak{a} < 1$, we have

$$\mathfrak{N}_1 < \left[\|\aleph\|_\infty \prod_{i=1}^{k} \|\lambda_i\|_{L^{p_i}_{\Delta}}\right]^{-1} \approx 0.2597173925.$$

Taking $\mathfrak{N}_1 = \frac{1}{4}$. In addition if we take

$$\Gamma_r = 10^{-4r}, \quad \Lambda_r = 10^{-(4r+3)},$$

then

$$\Gamma_{r+1} = 10^{-(4r+4)} < \frac{1}{5} \times 10^{-(4r+3)} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r = 10^{-(4r+3)} < \Gamma_r = 10^{-4r},$$

$\theta\Lambda_r = 24.00287746 \times 10^{-(4r+3)} < \frac{1}{4} \times 10^{-4r} = \mathfrak{N}_1\Gamma_r$, $r = 1, 2, 3, \ldots$, and $\mathsf{g}_1, \mathsf{g}_2$ satisfy the following growth conditions:

$$\begin{aligned}\mathsf{g}_1(\upsilon) = \mathsf{g}_2(\upsilon) &\le \mathfrak{N}_1\Gamma_r = \frac{1}{4} \times 10^{-4r}, \;\; \upsilon \in \left[0, 10^{-4r}\right], \\ \mathsf{g}_1(\upsilon) = \mathsf{g}_2(\upsilon) &\ge \theta\Lambda_r = 24.00287746 \times 10^{-(4r+3)}, \;\; \upsilon \in \left[\frac{1}{5} \times 10^{-(4r+3)}, 10^{-(4r+3)}\right].\end{aligned}$$

Then all the conditions of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the iterative boundary value problem (1.1) has infinitely many solutions $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]})\}_{r=1}^{\infty}$ such that $\upsilon_\ell^{[r]}(t) \ge 0$ on $[0,1]$, $\ell = 1, 2$ and $r \in \mathbb{N}$.

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ALMOST PERIODIC POSITIVE SOLUTIONS FOR A DELAYED NONLINEAR DENSITY DEPENDENT MORTALITY NICHOLSON'S BLOWFLIES MODEL ON TIME SCALES

K. R. PRASAD[1], M. KHUDDUSH[1], K. V. VIDYASAGAR[1], §

ABSTRACT. In this paper we discuss a nonlinear density dependent mortality Nicholson's blowflies equation with multiple pairs of time varying delays. By contraction mapping theorem, we derived the necessary conditions for the existence of almost periodic positive solutions and by selecting suitable Lyapunov functionnal we study global asymptotic stability of the addressed model. Finally, some numerical simulations are listed to show the validity of our methods.

Keywords: Time scale, Nicholson's blowflies model; almost periodic positive solution, global asymptotic stability.

AMS Subject Classification: 34K14, 39A30, 34N05.

1. INTRODUCTION

The delay differential equation

$$\vartheta'(t) = \alpha\vartheta(t) + \beta\vartheta(t-\tau)e^{-\gamma\vartheta(t-\tau)},\ t \in \mathbb{R}$$

describes a population of the Australian sheep blowfly proposed by Gurney [10] in 1980 and is agreed with the experimental data obtained by Nicholson [18] in 1954. Since this equation explains Nicholson blowfly more accurately, the model and it's modifications have been now refereed to as the Nicholson's blowflies model. The theory of Nicholson's blowflies model has made remarkable progress (see[6, 12, 17, 21] and references therein). Recently, Qian and Wang [22], studied a nonlinear density dependent mortality Nicholson's blowflies equation with multiple pairs of time-varying delays

$$\vartheta'(t) = a(t) + b(t)e^{-\vartheta(t)} + \sum_{j=1}^{m} \beta_j(t)\vartheta(t-h_j(t))e^{-\gamma_j(t)\vartheta(t-g_j(t))},\ t \in \mathbb{R}, \tag{1}$$

[1] Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, India - 530003.
e-mail: rajendra92@rediffmail.com; ORCID: https://orcid.org/0000-0001-8162-1391.
e-mail: khuddush89@gmail.com; ORCID: https://orcid.org/0000-0002-1236-8334.
e-mail: vidyavijaya08@gmail.com; ORCID: https://orcid.org/0000-0003-4532-8176.
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and by utilising differential inequality techniques and the fluctuation lemma, a delay-independent criterion was determined to ensure the global asymptotic stability of the model.

Many authors [1, 9] believe that the discrete time model governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Recently, Zhu [25] considered the following discrete delayed Ricker model with survival rate,

$$\vartheta(t+1) = \gamma(t)\vartheta(t) + \vartheta(\tau(t))e^{r(t)\left[1-\frac{\vartheta(\tau(t))}{k(t)}\right]},\ t \in \mathbb{Z}^+, \tag{2}$$

and established global attractivity, extreme stability, and the periodicity of the solution of the model.

The differential, difference and dynamic equations on time scales are three theories which play important role for modeling in the environment. Among them, the theory of dynamic equations on time scales is the most recent and was introduced by Stefan Hilger in his PhD thesis in 1988 with three main features: unification, extension and discretization. Since a time scale is any closed and nonempty subset of the real numbers set, so we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviors [4, 5, 23]. Moreover, basic results on this issue have been well documented in the articles [2, 3] and monographs of Bohner and Peterson [7, 8]. In the real world phenomena, since the almost periodic variation of the environment plays a crucial role in many biological and ecological dynamical systems and is more frequent and general than the periodic variation of the environment. In this paper we systematically unify the existence of almost periodic solutions for nonlinear density dependent mortality Nicholson's blowflies equation with multiple pairs of time varying delays modelled by ordinary differential equations and their discrete analogues in the form of difference equations and to extend these results to more general time scales. The concept of almost periodic time scales was proposed by Li and Wang [13]. Based on this concept, some works have been done (see [14, 15, 16, 19, 20]).

Motivated by aforementioned works, in this paper we study almost periodic positive solutions of a nonlinear density dependent mortality Nicholson's blowflies equation with multiple pairs of time-varying delays,

$$\vartheta^{\Delta}(t) = -a(t)\vartheta(t) + b(t)e^{-\vartheta(t)} + \sum_{\ell=1}^{n} \beta_\ell(t)\vartheta(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))}, \tag{3}$$

where $t \in \mathbb{T}$ ($\mathbb{T}$ is an arbitrary almost periodic time scale), $a(t)\vartheta(t) - b(t)e^{-\vartheta(t)}$ represents the death rate of the population, $\beta_\ell(t)\vartheta(t-h_\ell(t))\ e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))}$ describes the time dependent birth function which involves maturation delay $h_\ell(t)$ and incubation delay $g_\ell(t)$ and gains the reproduces at its maximum rate $\frac{1}{\gamma_\ell(t)}$, all parameter functions of (3) are nonnegative, bounded positive almost periodic functions, and $\ell \in \mathfrak{J} := \{1, 2, ..., n\}$. When $\mathbb{T} = \mathbb{Z}^+$, the model (3) is similar to the model (2).

For any bounded function $f(t)$, we denote $f^+ = \sup_{t\in\mathbb{T}} f(t)$, $f^- = \inf_{t\in\mathbb{T}} f(t)$.

We assume the following conditions are true throughout the paper:

(H_1) We assume that the bounded almost periodic functions $a(t)$, $b(t)$, $\beta_\ell(t)$, $g_\ell(t)$, $h_\ell(t)$ satisfy $0 < a^- \leq a(t) \leq a^+$, $0 < b^- \leq b(t) \leq b^+$, $0 < \beta_\ell^- \leq \beta_\ell(t) \leq \beta_\ell^+$, $0 < g_\ell^- \leq g_\ell(t) \leq g_\ell^+$, $0 < h_\ell^- \leq h_\ell(t) \leq h_\ell^+$ for $\ell = 1, 2, 3, \cdots, n$.

(H_2) The initial functions associated with equation (3) is given by

$$\vartheta(t;\varphi) = \varphi(t) \text{ for } t \in [-\varrho^*, 0]_{\mathbb{T}},\ \varrho^* = \max\left\{\max_{\ell\in\mathfrak{J}} g_\ell^+,\ \max_{\ell\in\mathfrak{J}} h_\ell^+\right\}$$

where $\varphi(\cdot)$ denotes a real-valued bounded and continuous functions defined on $[-\varrho^*, 0]_{\mathbb{T}}$.

Due to biological reasons of the model (3), positive solutions are only meaningful. So, we restrict our attention to positive solutions of equation (3).

2. Preliminaries

In this section, we introduce some definitions and state some preliminary results which are useful in the sequel.

Definition 2.1. [7] *A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to [0, \infty)$ are defined by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$ and $\mu(t) = \rho(t) - t$, respectively.*

- *In this definition we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.*
- *The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.*
- *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.*
- *A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.*
- *By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ other intervals can be defined similarly.*

Definition 2.2. [7] *A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$; $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive provided $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}^k$ The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ and the set of all positively regressive functions and rd-continuous functions will be denoted by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R})$.*

Definition 2.3. [7] *If p is regressive function, then the generalized exponential function e_p is defined by*

$$e_p(t, s) = \exp\Big\{\int_s^t \xi_{\mu(x)}(p(x))\Delta x\Big\}$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{Log(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Lemma 2.1. [7] *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions; then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$; *(ii)* $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;

(iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$; *(iv)* $(e_p(\cdot, s))^\Delta = p(t)e_p(t, s)$.

Lemma 2.2. [7] *Suppose that $p \in \mathcal{R}^+$, then*

(i) $e_p(t,s) > 0$ *for all* $t,s \in \mathbb{T}$;
(ii) *if* $p(t) \leq q(t)$ *for all* $t \geq s$, $t,s \in \mathbb{T}$, *then* $e_p(t,s) \leq e_q(t,s)$ *for all* $t \geq s$.

Lemma 2.3. [7] *If $p \in \mathcal{R}$ and $a,b,c \in \mathbb{T}$, then $[e_p(c,\cdot)]^\Delta = -p[e_p(c,\cdot)]^\sigma$, and*

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

Lemma 2.4. [7] *Let $p : \mathbb{T} \to \mathbb{R}$ be right-dense continuous and regressive, $a \in \mathbb{T}$ and $u_a \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$u^\Delta(t) = p(t)u(t) + f(t), \quad u(a) = u_a$$

is given by

$$u(t) = e_r(t,a)u_a + \int_a^t e_r(t,\sigma(s))f(s)\Delta s.$$

Lemma 2.5 ([7], Corollary 6.7, pp 257)**.** *Let $p \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$, $p(t) \geq 0$, $u(t) \in \mathcal{R}$ and $\alpha \in \mathbb{R}$. Then*

$$u(t) \leq \alpha + \int_{t_0}^t u(s)p(s)\Delta(s), \ \forall t \in \mathbb{T},$$

implies

$$u(t) \leq \alpha e_p(t,t_0), \ \forall t \in \mathbb{T}.$$

Definition 2.4. [13] *A time scale $\mathbb{T}$ is called an almost periodic time scale if*

$$\Pi := \{\tau \in \mathbb{R} : t + \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.5. [13] *Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T},\mathbb{R})$ is said to be almost periodic on $\mathbb{T}$, if, for any $\varepsilon > 0$, the set*

$$E(\varepsilon,f) = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\}$$

is relatively dense in $\mathbb{T}$; that is, for any $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains at least one $\tau \in E(\varepsilon,f)$ such that

$$|f(t+\tau) - f(t)| < \varepsilon, \ \forall t \in \mathbb{T}.$$

The set $E(\varepsilon,f)$ is called the $\varepsilon-$translation number of $f(t)$. We denote the set of all such functions by $AP(\mathbb{T})$.

Lemma 2.6. [13] *If $f \in C(\mathbb{T},\mathbb{R})$ is an almost periodic function, then f is bounded on $\mathbb{T}$.*

Lemma 2.7. [13] *If $f,g \in C(\mathbb{T},\mathbb{R})$ are almost periodic functions, then $f+g$, fg are also almost periodic.*

Definition 2.6. [24] *Let $\vartheta \in \mathbb{R}^m$ and $\mathcal{A}(t)$ be an $m \times m$ rd-continuous matrix on $\mathbb{T}$; the linear system*

$$\vartheta^\Delta(t) = \mathcal{A}(t)\vartheta(t), \ t \in \mathbb{T}, \tag{4}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants k, α, projection P, and the fundamental solution matrix $\vartheta(t)$ of (4) satisfying

$$|\vartheta(t)\mathcal{P}\vartheta^{-1}(\sigma(\tau))|_0 \leq ke_{\ominus\alpha}(t,\sigma(\tau)), \ \tau, t \in \mathbb{T}, \ t \geq \tau,$$

$$|\vartheta(t)(\mathcal{I} - \mathcal{P})\vartheta^{-1}(\sigma(s))|_0 \leq ke_{\ominus\alpha}(\sigma(\tau),t), \ \tau, t \in \mathbb{T}, \ t \leq \tau,$$

where $|\cdot|_0$ is a matrix norm on $\mathbb{T}$; that is, if $\mathcal{A} = (a_{ij})_{m\times m}$, then we can take $|\mathcal{A}|_0 = (\sum_{i=1}^m \sum_{j=1}^m |a_{ij}|^2)^{1/2}$.

Lemma 2.8. [13] *If the linear system* (4) *admits an exponential dichotomy, then the following system* $\vartheta^{\Delta}(t) = \mathcal{A}(t)\vartheta(t) + f(t),\ t \in \mathbb{T}$, *has a solution as follows:*

$$\vartheta(t) = \int_{-\infty}^{t} \vartheta(t)\mathcal{P}\vartheta^{-1}(\sigma(\tau))f(\tau)\Delta\tau - \int_{t}^{+\infty} \vartheta(t)(\mathcal{I}-\mathcal{P})\vartheta^{-1}(\sigma(\tau))f(\tau)\Delta\tau,$$

where $\vartheta(t)$ *is the fundamental solution matrix of* (4).

Lemma 2.9. [13] *Let* $\mathcal{A}(t)$ *be a regressive* $n \times n$ *matrix-valued function on* $\mathbb{T}$. *Let* $t_0 \in \mathbb{T}$ *and* $\vartheta_0 \in \mathbb{R}^n$, *then the initial value problem* $\vartheta^{\Delta}(t) = \mathcal{A}(t)\vartheta(t),\ \vartheta(t_0) = \vartheta_0$ *has a unique solution* $\vartheta(t) = e_{\mathcal{A}}(t,t_0)\vartheta_0$.

Lemma 2.10. [13] *Let* $d_i(t) > 0$ *be a function on* $\mathbb{T}$ *such that* $-d_i(t) \in \mathcal{R}^+$ *for all* $t \in \mathbb{T}$ *and* $\min\limits_{1\le i\le m}\left\{\inf\limits_{t\in\mathbb{T}} d_i(t)\right\} > 0$. *Then the linear system*

$$\vartheta^{\Delta}(t) = diag\big(-d_1(t), -d_2(t), \cdots, -d_m(t)\big)\vartheta(t)$$

admits an exponential dichotomy on $\mathbb{T}$.

3. Existence of the unique positive almost periodic solution

Let $\mathcal{B} = \{\vartheta(t) : \vartheta \in \mathcal{C}(\mathbb{T},\mathbb{R}),\ \vartheta(t)$ is almost periodic function$\}$ with norm

$$\|\vartheta\|_{\mathcal{B}} = \sup_{t\in\mathbb{T}} |\vartheta(t)|.$$

Then $\mathcal{B}$ is a Banach space.

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *hold. Let* $\mathfrak{M} > \mathfrak{m}$ *be two positive constants satisfy*

(i) $\mathfrak{M} = (\|\varphi\|_{\mathcal{B}} + b^*)e^+,\ b^* = \max\limits_{t\in[t_0,+\infty)_{\mathbb{T}}} \int_{t_0}^{t} b(\tau)\Delta\tau,\ e^+ = \max\limits_{t\in[t_0,+\infty)_{\mathbb{T}}} e^{\int_{t_0}^{t}\sum_{\ell=1}^{n}\beta_\ell(s)\Delta s}$.

(ii) $\dfrac{1}{a^+}\left[b^- e^{-\mathfrak{M}} + \sum\limits_{\ell=1}^{n} \beta_\ell^- e^{-\gamma_\ell^+\mathfrak{M}}\right] \ge \mathfrak{m} \ge \dfrac{1}{a^+}\sum\limits_{\ell=1}^{n}\beta_\ell^- e^{-\gamma_\ell^+\mathfrak{M}}$.

Then the solution $\vartheta(t) = \vartheta(t,t_0,\varphi) \ge 0$ *for all* $t \in [t_0, \eta(\varphi))_{\mathbb{T}}$, *of* (3) *satisfies*

$$\mathfrak{m} \le \vartheta(t) \le \mathfrak{M},\ t \in [t_0,+\infty)_{\mathbb{T}}.$$

Proof. Let $\vartheta(t) = \vartheta(t,t_0,\varphi)$ is a solution of (3) with the initial condition $\vartheta(t_0) = \varphi$, where $\varphi(\cdot)$ denotes a real-valued bounded and continuous functions defined on $[-\varrho^*, 0]_{\mathbb{T}}$. At first, we prove that $\vartheta(t) \le \mathfrak{M},\ t \in [t_0, \eta(\varphi))_{\mathbb{T}}$, where $[t_0, \eta(\varphi))_{\mathbb{T}}$ is the maximal right interval of existence of $\vartheta(t,t_0,\varphi)$. For all $t \in [t_0,\eta(\varphi))_{\mathbb{T}}$, let $\varpi(t) = \max\limits_{t_0-\varrho\le\tau\le t}\vartheta(\tau)$, we get

$$\begin{aligned}\vartheta^{\Delta}(t) =& -a(t)\vartheta(t) + b(t)e^{-\vartheta(t)} + \sum_{\ell=1}^{n}\beta_\ell(t)\vartheta(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))}\\ &\le b(t) + \sum_{\ell=1}^{n}\beta_\ell(t)\vartheta(t-h_\ell(t)) \le b(t) + \sum_{\ell=1}^{n}\beta_\ell(t)\varpi(t).\end{aligned}$$

So,

$$\begin{aligned}\vartheta(t) &\le \vartheta(t_0) + b^* + \int_{t_0}^{t}\left[\sum_{\ell=1}^{n}\beta_\ell(\tau)\varpi(\tau)\right]\Delta\tau\\ &\le \|\varphi\|_{\mathcal{B}} + b^* + \int_{t_0}^{t}\left[\sum_{\ell=1}^{n}\beta_\ell(\tau)\right]\varpi(\tau)\Delta\tau,\ t \in [t_0,\eta(\varphi))_{\mathbb{T}}.\end{aligned} \tag{5}$$

Since (5) is true for every $t \in [t_0, \eta(\varphi))_{\mathbb{T}}$ and $\varpi(t) = \max_{t_0-\varrho \le \tau \le t} \vartheta(\tau)$, it follows that

$$\varpi(t) \le \|\varphi\|_{\mathcal{B}} + b^* + \int_{t_0}^{t} \left[\sum_{\ell=1}^{n} \beta_\ell(\tau) \right] \varpi(\tau)\Delta\tau, \ t \in [t_0, \eta(\varphi))_{\mathbb{T}}.$$

Now by Lemma 2.5, we get

$$\vartheta(t) \le \varpi(t) \le (\|\varphi\|_{\mathcal{B}} + b^*) \exp\left\{ \int_{t_0}^{t} \left[\sum_{\ell=1}^{n} \beta_\ell(\tau) \right] \Delta\tau \right\}, \ t \in [t_0, \eta(\varphi))_{\mathbb{T}}.$$

Thus,

$$\vartheta(t) \le \mathfrak{M}, \ t \in [t_0, \eta(\varphi))_{\mathbb{T}}.$$

Next, we show that

$$\mathfrak{m} \le \vartheta(t), \ t \in [t_0, \eta(\varphi))_{\mathbb{T}}. \tag{6}$$

To prove this claim, we show that for any $\lambda < 1$, the following inequality holds

$$\vartheta(t) > \lambda\mathfrak{m}, \ t \in [t_0, \eta(\varphi))_{\mathbb{T}}. \tag{7}$$

By way of contradiction, assume that (7) does not hold. Then, there exists $t^* \in [t_0, \eta(\varphi))_{\mathbb{T}}$ such that

$$\vartheta(t^*) \le \lambda\mathfrak{m}, \ \vartheta(t) > \lambda, \ t \in [t_0 - \varrho, t^*)_{\mathbb{T}}.$$

Therefore, there must be a positive constant $\mu \le 1$ such that

$$\vartheta(t^*) = \lambda\mu\mathfrak{m}, \ \vartheta(t) > \lambda\mu, \ t \in [t_0 - \varrho, t^*)_{\mathbb{T}}.$$

Since $\lambda\mu < 1$, it follows that

$$\begin{aligned} 0 \ge \vartheta^{\Delta}(t^*) &= -a(t^*)\vartheta(t^*) + b(t^*)e^{-\vartheta(t^*)} + \sum_{\ell=1}^{n} \beta_\ell(t^*)\vartheta(t^* - h_\ell(t^*))e^{-\gamma_\ell(t^*)\vartheta(t^*-g_\ell(t^*))} \\ &\ge -a^+\lambda\mu\mathfrak{m} + b^-e^{-\mathfrak{M}} + \sum_{\ell=1}^{n} \beta_\ell^-\lambda\mu e^{-\gamma_\ell^+\mathfrak{M}} \ge b^-e^{-\mathfrak{M}} - \lambda\mu\left[a^+\mathfrak{m} - \sum_{\ell=1}^{n} \beta_\ell^- e^{-\gamma_\ell^+\mathfrak{M}} \right] \\ &\ge b^-e^{-\mathfrak{M}} - \left[a^+\mathfrak{m} - \sum_{\ell=1}^{n} \beta_\ell^- e^{-\gamma_\ell^+\mathfrak{M}} \right] > 0. \end{aligned}$$

Which is a contradiction and hence (7) holds. Letting $\lambda \to 1$, we get (6). Similar to the proof of Theorem 2.3.1 in [11], we can obtain that $\eta(\varphi) = +\infty$. Therefore,

$$\mathfrak{m} \le \vartheta(t) \le \mathfrak{M}, \ t \in [t_0, +\infty)_{\mathbb{T}}.$$

□

For $\varpi \in \mathcal{B}$, consider the equation

$$\vartheta^{\Delta}(t) = -a(t)\vartheta(t) + b(t)e^{-\varpi(t)} + \sum_{\ell=1}^{n} \beta_\ell(t)\varpi(t - h_\ell(t))e^{-\gamma_\ell(t)\varpi(t-g_\ell(t))}. \tag{8}$$

Since $\inf_{t\in\mathbb{T}} a(t) = a^- > 0$, then from Lemma 2.10 the linear equation $\vartheta^{\Delta}(t) = -a(t)\vartheta(t)$ admits exponential dichotomy on $\mathbb{T}$. Hence, by Lemma 2.8, the equation (8) has exactly one almost periodic solution,

$$\vartheta_\varpi(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(\tau)) \left[b(\tau)e^{-\varpi(\tau)} + \sum_{\ell=1}^{n} \beta_\ell(\tau)\varpi(\tau - h_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))} \right] \Delta\tau.$$

Define the operator $\aleph : \mathcal{B} \to \mathcal{B}$,

$$(\aleph\varpi)(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(\tau))\Big[b(\tau)e^{-\varpi(\tau)} + \sum_{\ell=1}^{n} \beta_\ell(\tau)\varpi(\tau - h_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\Big]\Delta\tau.$$

It is clear that, $\varpi(t)$ is the almost periodic solution of equation (3) if and only if ϖ is the fixed point of the operator $\aleph$.

For convenience, we take $M = \max\Big\{\sum_{\ell=1}^{n} \beta_\ell^{+}\|\varphi\|_{\mathcal{B}}e^{+},\, a^{+}\mathfrak{M}\Big\}$,

Theorem 3.2. *Suppose that the hypothesis of Theorem 3.1 satisfied. Then equation* (3) *has a unique almost periodic positive solution.*

Proof. It is clear from the Theorem 3.1 that $\aleph$ is self mapping on Ξ, where

$$\Xi = \big\{\varpi(t) \in \mathcal{B} : \mathfrak{m} \le \varpi(t) \le \mathfrak{M},\, t \in \mathbb{T}\big\}.$$

Next, we prove that $\aleph$ is a contraction mapping on Ξ. For $\vartheta, \varpi \in \Xi$, consider

$$\begin{aligned}
&\|\aleph\vartheta - \aleph\varpi\|_{\mathcal{B}} = \sup_{t\in\mathbb{T}} |(\aleph\vartheta)(t) - (\aleph\varpi)(t)| \\
&\le \sup_{t\in\mathbb{T}}\Bigg\{\int_{-\infty}^{t} e_{-a}(t,\sigma(\tau))\Big[-b(\tau)\Big(e^{-\vartheta(\tau)} - e^{-\varpi(\tau)}\Big) \\
&\quad + \sum_{\ell=1}^{n} \beta_\ell(\tau)\Big(\big[\vartheta(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - \varpi(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\big] \\
&\quad + \int_{\tau-g_\ell(\tau)}^{\tau-h_\ell(\tau)} \vartheta^{\Delta}(s)e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))}\Delta s - \int_{\tau-g_\ell(\tau)}^{\tau-h_\ell(\tau)} \varpi^{\Delta}(s)e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\Delta s\Big)\Big]\Delta\tau\Bigg\}
\end{aligned}$$

From Theorem 3.1, we note that

$$\vartheta^{\Delta}(t) \le \sum_{\ell=1}^{n} \beta_\ell(t)\varpi(t) \le \sum_{\ell=1}^{n} \beta_\ell(t)\|\varphi\|_{\mathcal{B}} \exp\left\{\int_{t_0}^{t}\sum_{\ell=1}^{n} \beta_\ell(s)\Delta s\right\} \le \sum_{\ell=1}^{n} \beta_\ell^{+}\|\varphi\|_{\mathcal{B}}e^{+},$$

and $\varpi^{\Delta}(t) \ge -a(t)\varpi(t) \ge -a^{+}\mathfrak{M}$. Therefore,

$$\begin{aligned}
&\|\aleph\vartheta - \aleph\varpi\|_{\mathcal{B}} \le \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} e_{-a}(t,\sigma(\tau))\Big[b^{+}\left|e^{-\vartheta(\tau)} - e^{-\varpi(\tau)}\right| \\
&\quad + \sum_{\ell=1}^{n} \beta_\ell^{+}\left|\vartheta(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - \varpi(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\right| \\
&+ \sum_{\ell=1}^{n} \beta_\ell^{+}\big[g_\ell(\tau) - h_\ell(\tau)\big]\left|\sum_{\ell=1}^{n} \beta_\ell^{+}\|\varphi\|_{\mathcal{B}}e^{+}e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - (d^{+} + a^{+}\mathfrak{M})e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\right|\Big]\Delta\tau \\
&\le \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} e_{-a}(t,\sigma(\tau))\Big[b^{+}\left|e^{-\vartheta(\tau)} - e^{-\varpi(\tau)}\right| \\
&\quad + \sum_{\ell=1}^{n} \beta_\ell^{+}\left|\vartheta(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - \varpi(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\right| \\
&\quad + \sum_{\ell=1}^{n} \beta_\ell^{+}\big[g_\ell(\tau) - h_\ell(\tau)\big]M\left|e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\right|\Big]\Delta\tau.
\end{aligned}$$

By mean value theorem, we have $\left|e^{-\vartheta(\tau)} - e^{-\varpi(\tau)}\right| \leq e^{-\xi_1}\left|\vartheta(\tau) - \varpi(\tau)\right|$ where ξ_1 lies between $\vartheta(\tau)$ and $\varpi(\tau)$,

$$\left|\vartheta(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - \varpi(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\right| \\ \leq (1-\gamma_\ell\xi_2)e^{-\gamma_\ell^-\xi_2}\left|\vartheta(\tau - g_\ell(\tau)) - \varpi(\tau - g_\ell(\tau))\right|,$$

where ξ_2 lies between $\vartheta(\tau - g_\ell(\tau))$ and $\varpi(\tau - g_\ell(\tau))$, and

$$\left|e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\right| \leq \gamma_\ell^+ e^{-\gamma_\ell^-\xi_3}\left|\vartheta(\tau - g_\ell(\tau)) - \varpi(\tau - g_\ell(\tau))\right|,$$

where ξ_3 lies between $\vartheta(\tau - g_\ell(\tau))$ and $\varpi(\tau - g_\ell(\tau))$.
Hence,

$$\begin{aligned}
\|\aleph\vartheta - \aleph\varpi\|_{\mathcal{B}} &\leq \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} e_{-a}(t,\sigma(\tau))\Big[b^+e^{-\xi_1}\left|\vartheta(\tau) - \varpi(\tau)\right| \\
&\quad + \sum_{\ell=1}^{n} \beta_\ell^+(1-\gamma_\ell\xi_2)e^{-\gamma_\ell^-\xi_2}\left|\vartheta(\tau - g_\ell(\tau)) - \varpi(\tau - g_\ell(\tau))\right| \\
&\quad + M\sum_{\ell=1}^{n} \beta_\ell^+\left[g_\ell(\tau) - h_\ell(\tau)\right]\gamma_\ell^+e^{-\gamma_\ell^-\xi_3}\left|\vartheta(\tau - g_\ell(\tau)) - \varpi(\tau - g_\ell(\tau))\right|\Big]\Delta\tau \\
&\leq \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} e_{-a}(t,\sigma(\tau))\Big[b^+e^{-\xi_1} + \sum_{\ell=1}^{n}\beta_\ell^+(1-\gamma_\ell^-\xi_2)e^{-\gamma_\ell^-\xi_2} \\
&\quad + M\sum_{\ell=1}^{n}\beta_\ell^+g_\ell^+\gamma_\ell^+e^{-\gamma_\ell^-\xi_3}\Big]\Delta\tau\,\|\vartheta-\varpi\|_{\mathcal{B}} \\
&\leq \frac{1}{a^-}\Big[b^+e^{-\xi_1} + \sum_{\ell=1}^{n}\beta_\ell^+(1-\gamma_\ell^-\xi_2)e^{-\gamma_\ell^-\xi_2} + M\sum_{\ell=1}^{n}\beta_\ell^+g_\ell^+\gamma_\ell^+e^{-\gamma_\ell^-\xi_3}\Big]\|\vartheta-\varpi\|_{\mathcal{B}}
\end{aligned}$$

Since $\frac{1}{a^-}\Big[b^+e^{-\xi_1} + \sum_{\ell=1}^{n}\beta_\ell^+(1-\gamma_\ell^-\xi_2)e^{-\gamma_\ell^-\xi_2} + M\sum_{\ell=1}^{n}\beta_\ell^+g_\ell^+\gamma_\ell^+e^{-\gamma_\ell^-\xi_3}\Big] < 1$, it follows that $\aleph$ is a contraction mapping. Thus, by the contraction mapping fixed point theorem, the operator $\aleph$ has a unique fixed point ϑ^* in Ξ. This implies that the equation (3) has a unique almost periodic positive solution $\vartheta^*(t)$ and $\mathfrak{m} \leq \vartheta^*(t) \leq \mathfrak{M}$. □

For convenience, we take

$$\Gamma = 2\Big[b^- - \sum_{\ell=1}^{n}\beta_\ell^+ - \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- - M\sum_{\ell=1}^{n}\beta_\ell^+g_\ell^+\gamma_\ell^+\Big] \\ \times\Big[b^+ + \sum_{\ell=1}^{n}\beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^{n}\beta_\ell^+g_\ell^+\gamma_\ell^+\Big]^{-2}.$$

Theorem 3.3. *Suppose that the hypothesis of the Theorem 3.2 is satisfied and for any $t_0 \in [-\varrho^*, +\infty)_{\mathbb{T}}$,*

$$\int_{t_0}^{t}(\Gamma - \mu(\tau))\Delta\tau \to +\infty \text{ as } t \to +\infty.$$

Then equation (3) *has unique globally asymptotically stable almost periodic positive solution.*

Proof. By Theorem 3.2, we know that (3) has a unique almost periodic positive solution $\vartheta^*(t)$, and $\mathfrak{m} \leq \vartheta^*(t) \leq \mathfrak{M}$. Suppose $\vartheta(t)$ is any arbitrary solution of (8) with initial function $\varphi(t) > 0$, $t \in [\varrho^*, 0]_{\mathbb{T}}$. Now we prove that $\vartheta^*(t)$ is globally asymptotically stable.

Let $\varpi(t) = \vartheta(t) - \vartheta^*(t)$ and define $\mathcal{V}(\varpi) = \varpi^2$. Then, we have

$$\begin{aligned}\mathcal{V}^{\Delta}(\varpi) &= 2\varpi(t)\varpi^{\Delta}(t) + \mu(t)(\varpi^{\Delta}(t))^2\\ &= 2\varpi\Big[- a(t)\big(\vartheta(t) - \vartheta^*(t)\big) + b(t)\left(e^{-\vartheta(t)} - e^{-\vartheta^*(t)}\right)\\ &\quad + \sum_{\ell=1}^{n} \beta_\ell(t)\left(\vartheta(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))} - \vartheta^*(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta^*(t-g_\ell(t))}\right)\Big]\\ &\quad + \mu(t)\Big[- a(t)\big(\vartheta(t) - \vartheta^*(t)\big) + b(t)\left(e^{-\vartheta(t)} - e^{-\vartheta^*(t)}\right)\\ &\quad + \sum_{\ell=1}^{n} \beta_\ell(t)\left(\vartheta(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))} - \vartheta^*(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta^*(t-g_\ell(t))}\right)\Big]^2\end{aligned}$$

Similar argument employed in Theorem 3.2 yields,

$$\begin{aligned}\mathcal{V}^{\Delta}(\varpi) =& 2\left[-b^- + \sum_{\ell=1}^{n}\beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^{n}\beta_\ell^+ g_\ell^+\gamma_\ell^+\right]\|\varpi\|_{\mathcal{B}}\\ &+ \mu(t)\left[b^+ + \sum_{\ell=1}^{n}\beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^{n}\beta_\ell^+ g_\ell^+\gamma_\ell^+\right]^2\|\varpi\|_{\mathcal{B}}\\ =& -\left[b^+ + \sum_{\ell=1}^{n}\beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^{n}\beta_\ell^+ g_\ell^+\gamma_\ell^+\right]^2\Big[\Gamma - \mu(t)\Big]\|\varpi\|_{\mathcal{B}}.\end{aligned}$$

Let $\Omega(\vartheta) = \left[b^+ + \sum_{\ell=1}^{n}\beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^{n}\beta_\ell^+ g_\ell^+\gamma_\ell^+\right]^2 \vartheta^2$, then

$$\mathcal{V}^{\Delta}(\varpi(t)) \leq -\Big[\Gamma - \mu(t)\Big]\Omega(\|\vartheta\|_{\mathcal{B}}).$$

Integrating from t_0 to t, we obtain

$$\mathcal{V}(\varpi(t)) \leq \mathcal{V}(\varpi(t_0)) - \int_{t_0}^{t}\Big[\Gamma - \mu(\tau)\Big]\Omega(\|\vartheta\|_{\mathcal{B}})\Delta\tau.$$

So, we get

$$\int_{t_0}^{t}\Big[\Gamma - \mu(\tau)\Big]\Omega(\|\vartheta\|_{\mathcal{B}})\Delta\tau \leq \mathcal{V}(\varpi(t_0)) - \mathcal{V}(\varpi(t)) < \mathcal{V}(\varpi(t_0)) < +\infty.$$

Since

$$\int_{t_0}^{t}(\Gamma - \mu(\tau))\Delta\tau \to +\infty \text{ as } t \to +\infty,$$

it follows that

$$\Omega(\|\vartheta\|_{\mathcal{B}}) \to 0, \ \textit{i.e.}, \ \|\vartheta(t) - \vartheta^*(t)\|_{\mathcal{B}} \to 0.$$

Hence, $\vartheta^*(t)$ is globally asymptotically stable. □

4. Examples

Example 4.1. *Consider following nonlinear density dependent mortality Nicholson's blowflies model for $\mathbb{T} = \mathbb{R}$.*

$$\left.\begin{aligned} \vartheta^{\Delta}(t) =& -(2+\sin(2t))\vartheta(t) + |\cos(t)|e^{-\vartheta(t)} \\ &+ |\sin(t)|\vartheta\left(t - 2e^{\cos(\sqrt{2}t)}\right) e^{-(2+\sin(t))\vartheta\left(t-(4+2e^{\sin(\sqrt{2}t)})\right)}, \\ \vartheta(0) =& \, 0.1. \end{aligned}\right\} \tag{9}$$

It is clear that (9) satisfies all the assumptions of Theorem 3.3. Therefore, equation (9) has a unique almost periodic positive solution $\vartheta^*(t)$ which is globally asymptotically stable. The numerical simulations in Fig. 1 strongly support the conclusion.

Example 4.2. *Consider following nonlinear density dependent mortality Nicholson's blowflies model for $\mathbb{T} = \mathbb{Z}^+$.*

$$\left.\begin{aligned} \vartheta(t+1) =& \, \vartheta(t) - (1+\cos t)\vartheta(t) + |\sin(t)|e^{-\vartheta(t)} \\ &+ |\sin(t)|\vartheta\left(t - e^{\sin(\sqrt{2}t)}\right) e^{-(2+\sin(t))\vartheta\left(t-(4+2e^{\cos(\sqrt{2}t)})\right)}, \\ \vartheta(0) =& \, 0.05. \end{aligned}\right\} \tag{10}$$

It is clear that (10) satisfies all the assumptions of Theorem 3.3. Therefore, equation (10) has a unique almost periodic positive solution $\vartheta^*(t)$ which is globally asymptotically stable. The numerical simulations in Fig. 2 strongly support the conclusion.

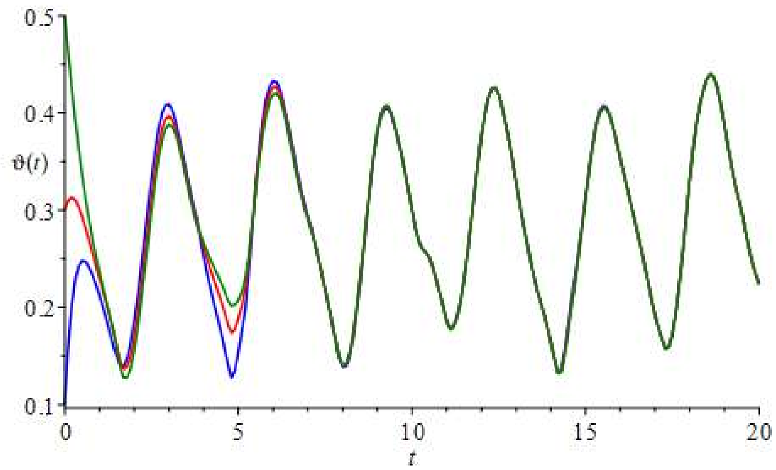

Figure 1. Numerical solution $\vartheta(t)$ of equation (9) for initial value $\varphi(\tau) = 0.1, 0.3, 0.5$ $\tau \in [-(4+2e), 0]$.

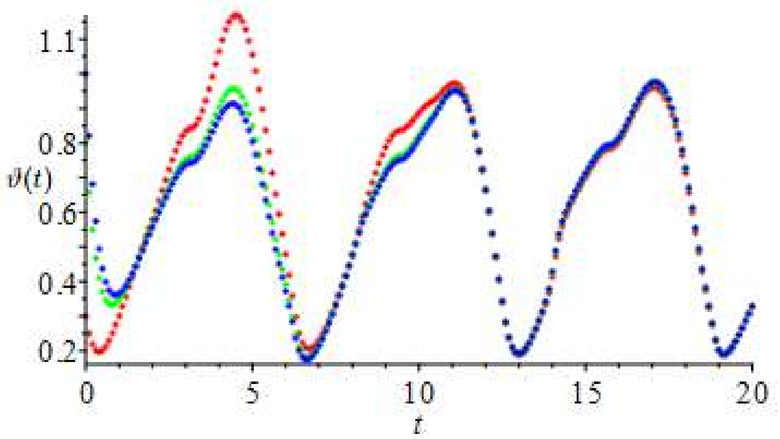

FIGURE 2. Numerical solution $\vartheta(t)$ of equation (9) for initial value $\varphi(\tau) = 0.3, 0.8, 1\ \tau \in [-(4+2e), 0]$.

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Kapula Rajendra Prasad received his M.Sc. and Ph.D. degrees from Andhra University, Visakhapatnam, India. Dr. Prasad did his post doctoral work at Auburn University, Auburn, USA. Currently, he is working as a professor and chairman-board of studies in the Department of Applied Mathematics, Andhra University, Visakhapatnam, India. His major research interest includes ODE, PDE, difference equations, dynamic equations on time scales, fractional order differential equations, mathematical modeling and fixed point theory.

Mahammad Khuddush received his M.Sc., M.Phil. and Ph.D. degrees from Andhra University, Visakhapatnam, India. His major research interest includes ODE, PDE, dynamic equations on time scales, fractional order differential equations, mathematical modeling and fixed point theory.

Kuparala Venkata Vidyasagar received his M.Sc. from Andhra University, Visakhapatnam, India. He is working as a lecturer in the Department of Mathematics, Government Degree College for Women, Marripalem, Narsipatman, India and also pursuing his Ph.D. under the guidance of Prof. K. Rajendra Prasad, Department of Applied Mathematics, Andhra University, Visakhapatnam, India. His research area is dynamic equations on time scales.

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DENUMERABLY MANY POSITIVE SOLUTIONS FOR ITERATIVE SYSTEM OF BOUNDARY VALUE PROBLEMS WITH N-SINGULARITIES ON TIME SCALES

K. R. PRASAD[1], MAHAMMAD KHUDDUSH[2], AND K. V. VIDYASAGAR[3]

Abstract. In this paper we consider a iterative system of two-point boundary value problems with integral boundary conditions having n singularities and involve an increasing homeomorphism, positive homomorphism operator. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of denumerably many positive solutions. Finally we provide an example to check validity of our obtained results.

1. Introduction

Theory of time scales was created to unify continuous and discrete analysis. Difference and differential equations can be studied simultaneously by studying dynamic equations on time scales. Since a time scale is any closed and nonempty subset of the real numbers set. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles [1, 2] and monographs of Bohner and Peterson [7, 8].

The study of turbulent flow through porous media is important for a wide range of scientific and engineering applications such as fluidized bed combustion, compact

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heat exchangers, combustion in an inert porous matrix, high temperature gas-cooled reactors, chemical catalytic reactors [9] and drying of different products such as iron ore [15]. To study such type of problems, Leibenson [13] introduced the following p-Laplacian equation

$$\Big(\varphi_p(\varpi'(t))\Big)' = f\Big(t, \varpi(t), \varpi'(t)\Big),$$

where $\varphi_p(\varpi) = |\varpi|^{p-2}\varpi$, $p > 1$, is the p-Laplacian operator its inverse function is denoted by $\varphi_q(\tau)$, with $\varphi_q(\tau) = |\tau|^{q-2}\tau$ and p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. It is well known fact that the p-Laplacian operator and fractional calculus arises from many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science, it is worth studying the fractional differential equations with p-Laplacian operator.

In this paper, we consider an operator φ called increasing homeomorphism and positive homomorphism operator (IHPHO), which generalizes and improves the p-Laplacian operator for some $p > 1$ and φ is not necessarily odd. Liang and Zhang [14] studied countably many positive solutions for nonlinear singular m–point boundary value problems on time scales with IHPHO,

$$\begin{aligned} &\Big(\varphi(\varpi^{\Delta}(t))\Big)^{\nabla} + a(t)f\Big(\varpi(t)\Big) = 0, \quad t \in [0, T]_{\mathbb{T}},\\ &\varpi(0) = \sum_{i=1}^{m-2} a_i\varpi(\xi_i), \quad \varpi^{\Delta}(T) = 0, \end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [10], Dogan considered second order p-boundary value problem on time scales,

$$\begin{aligned} &\Big(\varphi_p(\varpi^{\Delta}(t))\Big)^{\nabla} + \omega(t)f\Big(t, \varpi(t)\Big) = 0, \quad t \in [0, T]_{\mathbb{T}},\\ &\varpi(0) = \sum_{i=1}^{m-2} a_i\varpi(\xi_i), \quad \varphi_p(\varpi^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i\varphi_p(\varpi^{\Delta}(\xi_i)), \end{aligned}$$

and established existence of multiple positive solutions by applying fixed-point index theory.

Inspired by aforementioned works, in this paper by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of denumerably many positive solutions for dynamical iterative system of two-point boundary value problem with n singularities and involving IHPHO on time scales,

$$\left.\begin{aligned} &\varphi\Big(\varpi_j^{\Delta\nabla}(t)\Big) + \chi(t)f_j\Big(\varpi_{j+1}(t)\Big) = 0, \quad 1 \le j \le \ell, \quad t \in [0, 1]_{\mathbb{T}},\\ &\varpi_{\ell+1}(t) = \varpi_1(t), \quad t \in [0, 1]_{\mathbb{T}}, \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \alpha\varpi_j(0) - \beta\varpi_j^{\Delta}(0) &= \int_0^1 \kappa_1(\tau)\varpi_j(\tau)\nabla\tau, \quad 1 \le j \le \ell,\\ \gamma\varpi_j(1) + \delta\varpi_j^{\Delta}(1) &= \int_0^1 \kappa_2(\tau)\varpi_j(\tau)\nabla\tau, \quad 1 \le j \le \ell, \end{aligned}\right\} \tag{1.2}$$

where $\ell \in \mathbb{N}$, $\chi(t) = \prod_{i=1}^{n} \chi_i(t)$ and each $\chi_i(t) \in L^{p_i}_{\nabla}([0,1]_{\mathbb{T}})$, $p_i \geq 1$, has a singularity in the interval $\left(0, \frac{1}{2}\right)_{\mathbb{T}}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is an IHPHO with $\varphi(0) = 0$.

A projection $\varphi : \mathbb{R} \to \mathbb{R}$ is called a IHPHO, if the following three conditions are fulfilled:

(*a*) $\varphi(\tau_1) \leq \varphi(\tau_2)$ whenever $\tau_1 \leq \tau_2$, for any real numbers τ_1, τ_2;
(*b*) φ is a continuous bijection and its inverse φ^{-1} is continuous;
(*c*) $\varphi(\tau_1\tau_2) = \varphi(\tau_1)\varphi(\tau_2)$ for any real numbers τ_1, τ_2.

We use following notations in the entire paper: $i = 1, 2,\ \mathfrak{z} \in (0, 1/2)_{\mathbb{T}}$,

$$a(t) = \gamma + \delta - \gamma t, \quad b(t) = \beta + \alpha t, d = \alpha\gamma + \alpha\delta + \beta\gamma,$$

$$\aleph_0(t,\tau) = \frac{1}{d}\begin{cases} a(\tau)b(t), & t \leq \tau, \\ a(t)b(\tau), & \tau \leq t, \end{cases} \quad c_i = \int_0^1 \left[\int_0^1 \aleph_0(\tau_1, \tau_2)\kappa_i(\tau_1)\nabla\tau_1\right]\chi(\tau_2)\nabla\tau_2,$$

$$u_a = \frac{1}{d}\int_0^1 \kappa_1(\tau)a(\tau)\nabla\tau, \quad u_b = \frac{1}{d}\int_0^1 \kappa_1(\tau)b(\tau)\nabla\tau, \quad \kappa_i^* = \int_0^1 \kappa_i(\tau)\nabla\tau,$$

$$v_a = \frac{1}{d}\int_0^1 \kappa_2(\tau)a(\tau)\nabla\tau, \quad v_b = \frac{1}{d}\int_0^1 \kappa_2(\tau)b(\tau)\nabla\tau, \quad \kappa_i(\mathfrak{z}) = \int_{\mathfrak{z}}^{1-\mathfrak{z}} \kappa_i(\tau)\nabla\tau,$$

$$\eta(t) = \frac{(1-v_b)a(t) + v_a b(t)}{d[(1-u_a)(1-v_b) - u_b v_a]}, \quad \lambda(t) = \frac{(1-u_a)b(t) + u_b a(t)}{d[(1-u_a)(1-v_b) - u_b v_a]},$$

$$\eta^* = \max_{t\in[0,1]_{\mathbb{T}}} \eta(t), \quad \eta(\mathfrak{z}) = \max_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}} \eta(t), \quad \lambda^* = \max_{t\in[0,1]_{\mathbb{T}}} \lambda(t), \quad \lambda(\mathfrak{z}) = \max_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}} \lambda(t).$$

We assume the following conditions are true in the entire paper:

(H_1) $f_j : [0, +\infty) \to [0, +\infty)$ and $\kappa_1, \kappa_2 : [0, 1]_{\mathbb{T}} \to [0, +\infty)$ are continuous;
(H_2) there exists a sequence $\{t_r\}_{r=1}^{\infty}$ such that $0 < t_{r+1} < t_r < \frac{1}{2}$,

$$\lim_{r\to\infty} t_r = t^* < \frac{1}{2}, \quad \lim_{t\to t_r} \chi_i(t) = +\infty, \quad i = 1, 2, \ldots, n,\ r \in \mathbb{N},$$

and each $\chi_i(t)$ does not vanish identically on any subinterval of $[0, 1]_{\mathbb{T}}$. Moreover, there exists $\delta_i > 0$ such that

$$\delta_i < \varphi^{-1}(\chi_i(t)) < \infty \quad \text{a.e. on } [0,1]_{\mathbb{T}}, \quad i = 1, 2, \ldots, n.$$

2. Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions; for details, see [3–5, 7, 12, 17, 18].

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$ and $\mu(t) = \rho(t) - t$, respectively.

- The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$. Otherwise, $\mathbb{T}_k = \mathbb{T}$.

- If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.
- By an interval time scale, we mean the intersection of a real interval with a given time scale, i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ other intervals can be defined similarly.

Definition 2.2. Let μ_Δ and μ_∇ be the Lebesgue Δ-measure and the Lebesgue ∇-measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.

(i) If there exists $\Gamma_1 \subset A$ with $\mu_\Delta(\Gamma_1) = 0$ such that P holds on $A\backslash\Gamma_1$, then P is said to hold Δ-a.e. on A.

(ii) If there exists $\Gamma_2 \subset A$ with $\mu_\nabla(\Gamma_2) = 0$ such that P holds on $A\backslash\Gamma_2$, then P is said to hold ∇-a.e. on A.

Definition 2.3. Let $E \subset \mathbb{T}$ be a ∇-measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be ∇-measurable function. We say that f belongs to $L^p_\nabla(E)$ provided that either

$$\int_E |f|^p(s)\nabla s < \infty \quad \text{if } p \in \mathbb{R},$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M \quad \nabla\text{-a.e. on } E, \text{ if } p = +\infty.$$

Lemma 2.1. *Let $E \subset \mathbb{T}$ be a ∇-measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is a ∇-integrable on E, then*

$$\int_E f(s)\nabla s = \int_E f(s)ds + \sum_{i \in I_E} \big(t_i - \rho(t_i)\big) f(t_i),$$

where $I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}$, $I \subset \mathbb{N}$, is the set of all left-scattered points of $\mathbb{T}$.

Lemma 2.2. *For any $\varrho(t) \in C([0, 1]_{\mathbb{T}})$, the boundary value problem,*

$$-\varphi(\varpi_1^{\Delta\nabla}(t)) = \varrho(t), \quad t \in [0, 1]_{\mathbb{T}}, \tag{2.1}$$

$$\left.\begin{aligned} \alpha\varpi_1(0) - \beta\varpi_1^\Delta(0) &= \int_0^1 \kappa_1(\tau)\varpi_1(\tau)\nabla, \\ \gamma\varpi_1(1) + \delta\varpi_1^\Delta(1) &= \int_0^1 \kappa_2(\tau)\varpi_1(\tau)\nabla, \end{aligned}\right\} \tag{2.2}$$

has a unique solution

$$\varpi_1(t) = \int_0^1 \aleph(t, \tau)\varphi^{-1}(\varrho(\tau))\nabla\tau,$$

where

$$\aleph(t,\tau) = \aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1.$$

Proof. Suppose ϖ_1 is a solution of (2.1), then

$$\begin{aligned}\varpi_1(t) &= -\int_0^t\int_0^\tau \varphi^{-1}(\varrho(\tau_1))\nabla\tau_1\Delta\tau + At + B\\ &= -\int_0^t (t-\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau + A_1 t + A_2,\end{aligned}$$

where $A_1 = \varpi_1^\Delta(0)$ and $A_2 = \varpi_1(0)$. By the conditions (2.2), we get

$$A_1 = \frac{1}{d}\int_0^1 [\alpha\kappa_2(\tau) - \gamma\kappa_1(\tau)]\vartheta_1(\tau)\nabla\tau + \frac{1}{d}\int_0^1 \alpha[\gamma(1-\tau)+\delta]\varphi^{-1}(\varrho(\tau))\nabla\tau$$

and

$$A_2 = \frac{1}{d}\int_0^1 [(\gamma+\delta)\kappa_1(\tau) + \beta\kappa_2(\tau)]\vartheta_1(\tau)\nabla\tau + \frac{1}{d}\int_0^1 \beta[\gamma(1-\tau)+\delta]\varphi^{-1}(\varrho(\tau))\nabla\tau.$$

So, we have

(2.3)

$$\varpi_1(t) = \int_0^1 \aleph_0(t,\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau + \frac{a(t)}{d}\int_0^1 \kappa_1(\tau)\vartheta_1(\tau)\nabla\tau + \frac{b(t)}{d}\int_0^1 \kappa_2(\tau)\vartheta_1(\tau)\nabla\tau.$$

By simple computations, we find that

(2.4)
$$\int_0^1 \kappa_1(\tau)\vartheta_1(\tau)\nabla\tau = \frac{c_1(1-v_b)+c_2u_b}{(1-u_a)(1-v_b)-u_bv_a},$$

(2.5)
$$\int_0^1 \kappa_2(\tau)\vartheta_1(\tau)\nabla\tau = \frac{c_2(1-u_a)+c_1v_a}{(1-u_a)(1-v_b)-u_bv_a}.$$

Plugging (2.4) and (2.5) into (2.3), we received

$$\begin{aligned}\varpi_1(t) &= \int_0^1 \aleph_0(t,\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau + c_1\eta(t) + c_2\lambda(t)\\ &= \int_0^1 \left[\aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1\right]\\ &\quad\times\varphi^{-1}(\varrho(\tau))\nabla\tau\\ &= \int_0^1 \aleph(t,\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau.\end{aligned}$$

This completes the proof. □

Lemma 2.3. *Suppose* (H_1)*-*(H_2) *hold. For* $\mathfrak{z}\in(0,\frac{1}{2})_{\mathbb{T}}$, *let*

$$\mathcal{L}(\mathfrak{z}) = \min\left\{\frac{\alpha\mathfrak{z}+\beta}{\alpha+\beta}, \frac{\gamma\mathfrak{z}+\delta}{\gamma+\delta}\right\} < 1.$$

Then $\aleph_0(t,\tau)$ *have the following properties:*

(i) $0 \le \aleph_0(t,\tau) \le \aleph_0(\tau,\tau)$ *for all* $t,\tau\in[0,1]_{\mathbb{T}}$*;*

(ii) $\mathcal{L}(\mathfrak{z})\aleph_0(\tau,\tau) \le \aleph_0(t,\tau)$ *for all* $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$.

Proof. (i) is evident. We establish (ii), for this, let $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ and $t \le \tau$. Then

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{b(t)}{b(\tau)} = \frac{\alpha t + \beta}{\alpha\tau + \beta} \ge \frac{\alpha\mathfrak{z} + \beta}{\alpha + \beta} \ge \mathcal{L}(\mathfrak{z}).$$

For $\tau \le t$,

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{a(t)}{a(\tau)} = \frac{\gamma + \delta - \gamma t}{\gamma + \delta - \gamma\mathfrak{z}} \ge \frac{\gamma\mathfrak{z} + \delta}{\gamma + \delta} \ge \mathcal{L}(\mathfrak{z}).$$

This completes the proof. □

Lemma 2.4. *Suppose* (H_1)*-*(H_2) *hold. Then* $\aleph(t,\tau)$ *satisfies properties:*

(i) $0 \le \aleph(t,\tau) \le \Xi\,\aleph_0(\tau,\tau)$ *for all* $t, \tau \in\in [0,1]_{\mathbb{T}}$*;*

(ii) $0 \le \Xi_{\mathfrak{z}}\aleph_0(\tau,\tau) \le \aleph(t,\tau)$ *for all* $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$, *where*

$$\Xi = 1 + \eta^*\kappa_1^* + \lambda^*\kappa_2^*$$

and

$$\Xi_{\mathfrak{z}} = \mathcal{L}(\mathfrak{z})\Big[1 + \eta(\mathfrak{z})\kappa_1(\mathfrak{z}) + \lambda(\mathfrak{z})\kappa_2(\mathfrak{z})\Big].$$

Proof. From Lemma 2.3, we get

$$\begin{aligned}
\aleph(t,\tau) &= \aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1\\
&\le \aleph_0(\tau,\tau) + \eta(t)\int_0^1 \aleph_0(\tau,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau,\tau)\kappa_2(\tau_1)\nabla\tau_1\\
&\le \left[1 + \eta(t)\int_0^1 \kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \kappa_2(\tau_1)\nabla\tau_1\right]\aleph_0(\tau,\tau)\\
&\le \Big[1 + \eta^*\kappa_1^* + \lambda^*\kappa_2^*\Big]\aleph_0(\tau,\tau).
\end{aligned}$$

On the other hand, for $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ and $\tau \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}
\aleph(t,\tau) &= \aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1\\
&\ge \aleph_0(t,\tau) + \eta(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1\\
&\ge \mathcal{L}(\mathfrak{z})\left[1 + \eta(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \kappa_2(\tau_1)\nabla\tau_1\right]\aleph_0(\tau,\tau)\\
&\ge \mathcal{L}(\mathfrak{z})\Big[1 + \eta^{**}\kappa_1^{**} + \lambda^{**}\kappa_2^{**}\Big]\aleph_0(\tau,\tau).
\end{aligned}$$

This completes the proof. □

Notice that an ℓ-tuple $(\varpi_1(t), \varpi_2(t), \varpi_3(t), \dots, \varpi_\ell(t))$ is a solution of the iterative boundary value problem (1.1)–(1.2) if and only if

$$\begin{aligned}
&\varpi_j(t) = \int_0^1 \aleph(t,\tau)\varphi^{-1}\Big[\chi(\tau)f_j(\varpi_{j+1}(\tau))\Big]\nabla\tau, \quad t \in [0,1]_{\mathbb{T}},\ 1 \le j \le \ell,\\
&\varpi_{\ell+1}(t) = \varpi_1(t), \quad t \in [0,1]_{\mathbb{T}},
\end{aligned}$$

i.e.,

$$\begin{aligned}\varpi_1(t) =& \int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\ &\times \varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

Let $\mathtt{B}$ be the Banach space $C_{ld}([0,1]_{\mathbb{T}},\mathbb{R})$ with the norm $\|\varpi\| = \max_{t\in[0,1]_{\mathbb{T}}}|\varpi(t)|$. For $\mathfrak{z}\in\left(0,\frac{1}{2}\right)$, we define the cone $\mathtt{K}_{\mathfrak{z}}\subset\mathtt{B}$ as

$$\mathtt{K}_{\mathfrak{z}} = \left\{\varpi\in\mathtt{B} : \varpi(t) \text{ is nonnegative and } \min_{t\in[\mathfrak{z},\,1-\mathfrak{z}]_{\mathbb{T}}}\varpi(t) \geq \frac{\Xi_{\mathfrak{z}}}{\Xi}\|\varpi(t)\|\right\}.$$

For any $\varpi_1\in\mathtt{K}_{\mathfrak{z}}$, define an operator $\Omega:\mathtt{K}_{\mathfrak{z}}\to\mathtt{B}$ by

$$\begin{aligned}(\Omega\varpi_1)(t) =& \int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\ &\times \varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

Lemma 2.5. *Assume that* (H_1)*-*(H_2) *hold. Then for each* $\mathfrak{z}\in\left(0,\frac{1}{2}\right)$, $\Omega(\mathtt{K}_{\mathfrak{z}})\subset\mathtt{K}_{\mathfrak{z}}$ *and* $\Omega:\mathtt{K}_{\mathfrak{z}}\to\mathtt{K}_{\mathfrak{z}}$ *is completely continuous.*

Proof. From Lemma 2.3, $\aleph(t,\tau)\geq 0$ for all $t,\tau\in[0,1]_{\mathbb{T}}$. So, $(\Omega\varpi_1)(t)\geq 0$. Also, for $\varpi_1\in\mathtt{K}$, we have

$$\begin{aligned}(\Omega\varpi_1)(t) \leq& \,\Xi\int_0^1 \aleph_0(\tau_1,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)\\ &\times f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

So,

$$\begin{aligned}\|\Omega\varpi_1\| \leq& \,\Xi\int_0^1 \aleph_0(\tau_1,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)\\ &\times f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\end{aligned}$$

$$\times f_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\bigg]\nabla\tau_2\bigg]\nabla\tau_1.$$

Again from Lemma 2.3, we get

$$\begin{aligned}&\min_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\varpi_1)(t)\}\\ \geq&\Xi_{\mathfrak{z}}\int_0^1 \aleph_0(\tau_1,\tau_1)\varphi^{-1}\bigg[\chi(\tau_1)f_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\bigg[\chi(\tau_2)\\ &\times f_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\varphi^{-1}\bigg[\chi(\tau_3)f_3\bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\bigg]\nabla\tau_2\bigg]\nabla\tau_1.\end{aligned}$$

It follows from the above two inequalities that

$$\min_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\varpi_1)(t)\}\geq\frac{\Xi_{\mathfrak{z}}}{\Xi}\|\Omega\varpi_1\|.$$

So, $\Omega\varpi_1\in\mathtt{K}_{\mathfrak{z}}$ and thus $\Omega(\mathtt{K}_{\mathfrak{z}})\subset\mathtt{K}_{\mathfrak{z}}$. Next, by standard methods and Arzela-Ascoli theorem, it can be proved easily that the operator Ω is completely continuous. The proof is complete. □

3. Denumerably Many Positive Solutions

For the existence of denumerably many positive solutions for iterative system of boundary value problem (1.1), we apply following theorems.

Theorem 3.1 ([11]). *Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and $\mathtt{M}_1$, $\mathtt{M}_2$ are open sets with $0\in\mathtt{M}_1,\overline{\mathtt{M}}_1\subset\mathtt{M}_2$. Let $\mathcal{A}:\mathcal{E}\cap(\overline{\mathtt{M}}_2\backslash\mathtt{M}_1)\to\mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{A}z\|\leq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_1$, *and* $\|\mathcal{A}z\|\geq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_2$, *or*
(b) $\|\mathcal{A}z\|\geq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_1$, *and* $\|\mathcal{A}z\|\leq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_2$.

Then $\mathcal{A}$ has a fixed point in $\mathcal{E}\cap(\overline{\mathtt{M}}_2\backslash\mathtt{M}_1)$.

Theorem 3.2 ([8, 16]). *Let $f\in L^p_\nabla(J)$, with $p>1$, $g\in L^q_\nabla(J)$, with $q>1$, and $\frac{1}{p}+\frac{1}{q}=1$. Then $fg\in L^1_\nabla(J)$ and $\|fg\|_{L^1_\nabla}\leq\|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$, where*

$$\|f\|_{L^p_\nabla}:=\begin{cases}\Big[\int_J|f|^p(s)\nabla s\Big]^{\frac{1}{p}}, & p\in\mathbb{R},\\ \inf\Big\{M\in\mathbb{R}\,/\,|f|\leq M\ \nabla\text{-a.e. on } J\Big\}, & p=\infty,\end{cases}$$

and $J=(a,b]_{\mathbb{T}}$.

Theorem 3.3 (Hölder's). *Let $f \in L_{\nabla}^{p_i}(J)$, with $p_i > 1$, for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then $\prod_{i=1}^{n} f_i \in L_{\nabla}^{1}(J)$ and*

$$\left\| \prod_{i=1}^{n} f_i \right\|_1 \leq \prod_{i=1}^{n} \|f_i\|_{p_i}.$$

Further, if $f \in L_{\nabla}^{1}(J)$ and $g \in L_{\nabla}^{\infty}(J)$, then $fg \in L_{\nabla}^{1}(J)$ and $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.

Consider the following three possible cases for $\chi_i \in L_{\nabla}^{p_i}([0,1]_{\mathbb{T}})$:

(i) $\sum_{i=1}^{n} \frac{1}{p_i} < 1$;
(ii) $\sum_{i=1}^{n} \frac{1}{p_i} = 1$;
(iii) $\sum_{i=1}^{n} \frac{1}{p_i} > 1$.

Firstly, we seek denumerably many positive solutions for the case $\sum_{i=1}^{n} \frac{1}{p_i} < 1$.

Theorem 3.4. *Suppose (H_1)-(H_3) hold, let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be a sequence with $\mathfrak{z}_r \in (t_{r+1}, t_r)$. Let $\{\Gamma_r\}_{r=1}^{\infty}$ and $\{\Theta_r\}_{r=1}^{\infty}$ be such that*

$$\Gamma_{r+1} < \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r < \Theta_r < \mathfrak{Z}\Theta_r < \Gamma_r, \quad r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{ \left[\Xi_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau \right]^{-1},\ 1 \right\}.$$

Assume that f satisfies

(C_1) *$f_j(\varpi) \leq \varphi(\mathfrak{N}_1\Gamma_r)$ for all $t \in [0,1]_{\mathbb{T}}$, $0 \leq \varpi \leq \Gamma_r$, where*

$$\mathfrak{N}_1 < \left[\Xi \, \|\aleph_0\|_{L_{\nabla}^{q}} \prod_{i=1}^{n} \left\| \varphi^{-1}(\chi_i) \right\|_{L_{\nabla}^{p_i}} \right]^{-1};$$

(C_2) *$f_j(\varpi) \geq \varphi(\mathfrak{Z}\Theta_r)$ for all $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, $\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r \leq \varpi \leq \Theta_r$.*

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \ldots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ such that $\varpi_j^{[r]}(t) \geq 0$ on $[0,1]_{\mathbb{T}}$, $j = 1, 2, \ldots, \ell$ and $r \in \mathbb{N}$.*

Proof. Let

$$\mathtt{M}_{1,r} = \{\varpi \in \mathtt{B} : \|\varpi\| < \Gamma_r\},$$

$$\mathtt{M}_{2,r} = \{\varpi \in \mathtt{B} : \|\varpi\| < \Theta_r\},$$

be open subsets of $\mathtt{B}$. Let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^* < t_{r+1} < \mathfrak{z}_r < t_r < \frac{1}{2},$$

for all $r \in \mathbb{N}$. For each $r \in \mathbb{N}$, we define the cone $\mathtt{K}_{\mathfrak{z}_r}$ by

$$\mathtt{K}_{\mathfrak{z}_r} = \left\{ \varpi \in \mathtt{B} : \varpi(t) \geq 0, \min_{t \in [\mathfrak{z}_r,\, 1-\mathfrak{z}_r]_{\mathbb{T}}} \varpi(t) \geq \frac{\Xi_{\mathfrak{z}_r}}{\Xi} \|\varpi(t)\| \right\}.$$

Let $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial \mathtt{M}_{1,r}$. Then $\varpi_1(\tau) \leq \Gamma_r = \|\varpi_1\|$ for all $\tau \in [0,1]_{\mathbb{T}}$. By (C_1) and for $\tau_{\ell-1} \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell &\leq \Xi\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)\Big]\nabla\tau_\ell\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\prod_{i=1}^n\chi_i(\tau_\ell)\Big]\nabla\tau_\ell\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^n\varphi^{-1}(\chi_i(\tau_\ell))\nabla\tau_\ell.\end{aligned}$$

There exists a $q>1$ such that $\frac{1}{q}+\sum_{i=1}^n\frac{1}{p_i}=1$. So,

$$\begin{aligned}\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell &\leq \Xi\,\mathfrak{N}_1\Gamma_r\Big\|\aleph_0\Big\|_{L^q_\nabla}\Big\|\prod_{i=1}^n\varphi^{-1}(\chi_i)\Big\|_{L^{p_i}_\nabla}\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^n\Big\|\varphi^{-1}(\chi_i)\Big\|_{L^{p_i}_\nabla}\\ &\leq \Gamma_r.\end{aligned}$$

It follows in similar manner for $\tau_{\ell-2} \in [0,1]_{\mathbb{T}}$ that

$$\begin{aligned}&\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\Bigg[\chi(\tau_{\ell-1})f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\Bigg]\nabla\tau_{\ell-1}\\ \leq&\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1}\\ \leq&\Xi\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1}\\ \leq&\Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})\Big]\nabla\tau_{\ell-1}\\ \leq&\Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Big[\prod_{i=1}^n\chi_i(\tau_{\ell-1})\Big]\nabla\tau_{\ell-1}\\ \leq&\Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\prod_{i=1}^n\varphi^{-1}(\chi_i(\tau_{\ell-1}))\nabla\tau_{\ell-1}\\ \leq&\Xi\,\mathfrak{N}_1\Gamma_r\|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^n\Big\|\varphi^{-1}(\chi_i)\Big\|_{L^{p_i}_\nabla}\\ \leq&\Gamma_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$(\Omega\varpi_1)(t)=\int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)$$

$$\times \varphi^{-1}\Big[\chi(\tau_3)f_3\Big(\int_0^1 \aleph(\tau_3,\tau_4)\cdots$$
$$\times f_{\ell-1}\Big(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\big]\nabla\tau_\ell\Big)\cdots\nabla\tau_3\Big]\nabla\tau_2\Big]\nabla\tau_1$$
$$\leq \Gamma_r.$$

Since $\Gamma_r = \|\varpi_1\|$ for $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{1,r}$ we get

$$\|\Omega\varpi_1\| \leq \|\varpi_1\|. \tag{3.1}$$

Let $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$. Then

$$\Theta_r = \|\varpi_1\| \geq \varpi_1(t) \geq \min_{t\in[\mathfrak{z}_r,1-\mathfrak{z}_r]_{\mathbb{T}}} \varpi_1(t) \geq \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\|\varpi_1\| \geq \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r.$$

By (C_2) and for $\tau_{\ell-1} \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\big]\nabla\tau_\ell &\geq \Xi_{\mathfrak{z}_r}\int_{\mathfrak{z}_r}^{1-\mathfrak{z}_r} \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\big]\nabla\tau_\ell\\
&\geq \Xi_{\mathfrak{z}_r}\mathfrak{Z}\Theta_r\int_{\mathfrak{z}_r}^{1-\mathfrak{z}_r} \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}(\chi(\tau_\ell))\nabla\tau_\ell\\
&\geq \Xi_{\mathfrak{z}_r}\mathfrak{Z}\Theta_r\int_{\mathfrak{z}_r}^{1-\mathfrak{z}_r} \aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^n\varphi^{-1}(\chi_i(\tau_\ell))\nabla\tau_\ell\\
&\geq \Xi_{\mathfrak{z}_1}\mathfrak{Z}\Theta_r\prod_{i=1}^n\delta_i\int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau_\ell,\tau_\ell)\nabla\tau_\ell\\
&\geq \Theta_r.\end{aligned}$$

Continuing with bootstrapping argument we get

$$(\Omega\varpi_1)(t) = \int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Big[\chi(\tau_1)f_1\Big(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Big[\chi(\tau_2)f_2\Big(\int_0^1 \aleph(\tau_2,\tau_3)$$
$$\times \varphi^{-1}\Big[\chi(\tau_3)f_3\Big(\int_0^1 \aleph(\tau_3,\tau_4)\cdots$$
$$\times f_{\ell-1}\Big(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\big]\nabla\tau_\ell\Big)\cdots\nabla\tau_3\Big]\nabla\tau_2\Big]\nabla\tau_1$$
$$\geq \Theta_r.$$

Thus, if $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{K}_{2,r}$, then

$$\|\Omega\varpi_1\| \geq \|\varpi_1\|. \tag{3.2}$$

It is evident that $0 \in \mathtt{M}_{2,k} \subset \overline{\mathtt{M}}_{2,k} \subset \mathtt{M}_{1,k}$. From (3.1) and (3.2), it follows from Theorem 3.1 that the operator Ω has a fixed point $\varpi_1^{[r]} \in \mathtt{K}_{\mathfrak{z}_r} \cap \Big(\overline{\mathtt{M}}_{1,r}\backslash\mathtt{M}_{2,r}\Big)$ such that $\varpi_1^{[r]}(t) \geq 0$

on $[0,1]_{\mathbb{T}}$, and $r \in \mathbb{N}$. Next setting $\varpi_{\ell+1} = \varpi_1$, we obtain denumerably many positive solutions $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \ldots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ of (1.1)–(1.2) given iteratively by

$$\varpi_j(t) = \int_0^1 \aleph(t,\tau)\varphi^{-1}\Big[\chi(\tau) f_j(\varpi_{j+1}(\tau))\Big]\nabla\tau, \quad t \in [0,1]_{\mathbb{T}},\ j = \ell, \ell-1, \ldots, 1.$$

The proof is completed. □

For $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, we have the following theorem.

Theorem 3.5. *Suppose (H_1)-(H_3) hold, let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be a sequence with $\mathfrak{z}_r \in (t_{r+1}, t_r)$. Let $\{\Gamma_r\}_{r=1}^{\infty}$ and $\{\Theta_r\}_{r=1}^{\infty}$ be such that*

$$\Gamma_{r+1} < \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r < \Theta_r < \mathfrak{Z}\Theta_r < \Gamma_r, \quad r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{ \left[\Xi_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau\right]^{-1},\ 1\right\}.$$

Assume that f satisfies

(C_3) *$f_j(\varpi) \le \varphi(\mathfrak{N}_2\Gamma_r)$ for all $t \in [0,1]_{\mathbb{T}}$, $0 \le \varpi \le \Gamma_r$, where*

$$\mathfrak{N}_2 < \min\left\{ \left[\Xi\, \|\aleph_0\|_{L^\infty_\nabla} \prod_{i=1}^{n} \left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla}\right]^{-1}, \mathfrak{Z}\right\};$$

(C_4) *$f_j(\varpi) \ge \varphi(\mathfrak{Z}\Theta_r)$ for all $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, $\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r \le \varpi \le \Theta_r$.*

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \ldots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ such that $\varpi_j^{[r]}(t) \ge 0$ on $[0,1]_{\mathbb{T}}$, $j = 1,2,\ldots,\ell$, and $r \in \mathbb{N}$.*

Proof. For a fixed r, let $\mathtt{M}_{1,r}$ be as in the proof of Theorem 3.4 and let $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{2,r}$. Again $\varpi_1(\tau) \le \Gamma_r = \|\varpi_1\|$ for all $\tau \in [0,1]_{\mathbb{T}}$. By (C_3) and for $\tau_{\ell-1} \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}
\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell &\le \Xi \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell \\
&\le \Xi\, \mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)\Big]\nabla\tau_\ell \\
&\le \Xi\, \mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\left[\prod_{i=1}^{n}\chi_i(\tau_\ell)\right]\nabla\tau_\ell \\
&\le \Xi\, \mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^{n}\varphi^{-1}(\chi_i(\tau_\ell))\nabla\tau_\ell \\
&\le \Xi\, \mathfrak{N}_2\Gamma_r \|\aleph_0\|_{L^\infty_\nabla} \prod_{i=1}^{n} \left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla} \\
&\le \Gamma_r.
\end{aligned}$$

It follows in similar manner for $\tau_{\ell-2} \in [0,1]_{\mathbb{T}}$ that

$$\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\left[\chi(\tau_{\ell-1}) f_{\ell-1}\left(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\right)\right]\nabla\tau_{\ell-1}$$

$$
\begin{aligned}
&\leq \int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1}\\
&\leq \Xi\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1}\\
&\leq \Xi\,\mathfrak{N}_2\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})\Big]\nabla\tau_{\ell-1}\\
&\leq \Xi\,\mathfrak{N}_2\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\left[\prod_{i=1}^n\chi_i(\tau_{\ell-1})\right]\nabla\tau_{\ell-1}\\
&\leq \Xi\,\mathfrak{N}_2\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\prod_{i=1}^n\varphi^{-1}(\chi_i(\tau_{\ell-1}))\nabla\tau_{\ell-1}\\
&\leq \Xi\,\mathfrak{N}_2\Gamma_r\|\aleph_0\|_{L^\infty_\nabla}\prod_{i=1}^n\left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla}\\
&\leq \Gamma_r.
\end{aligned}
$$

Continuing with this bootstrapping argument, we get

$$
\begin{aligned}
(\Omega\varpi_1)(t)=&\int_0^1\aleph(t,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1\aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)f_2\Bigg(\int_0^1\aleph(\tau_2,\tau_3)\\
&\times\varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1\aleph(\tau_3,\tau_4)\cdots\\
&\times f_{\ell-1}\Bigg(\int_0^1\aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1\\
\leq&\,\Gamma_r.
\end{aligned}
$$

Since $\Gamma_r=\|\varpi_1\|$ for $\varpi_1\in \mathtt{K}_{\mathfrak{z}_r}\cap\partial\mathtt{M}_{1,r}$, we get $\|\Omega\varpi_1\|\leq\|\varpi_1\|$. Now define $\mathtt{M}_{2,r}=\{\varpi_1\in \mathtt{B}:\|\varpi_1\|<\Theta_r\}$. Let $\varpi_1\in\mathtt{K}_{\mathfrak{z}_r}\cap\partial\mathtt{M}_{2,r}$ and let $\tau\in[\mathfrak{z}_r,1-\mathfrak{z}_r]_{\mathbb{T}}$. Then the argument leading to (3.2) can be done to the present case. Hence, the theorem is proved. □

Lastly, the case $\sum_{i=1}^n\frac{1}{p_i}>1$.

Theorem 3.6. *Suppose* (H_1)-(H_2) *hold, let* $\{\mathfrak{z}_r\}_{r=1}^\infty$ *be a sequence with* $\mathfrak{z}_r\in(t_{r+1},t_r)$. *Let* $\{\Gamma_r\}_{r=1}^\infty$ *and* $\{\Theta_r\}_{r=1}^\infty$ *be such that*

$$\Gamma_{r+1}<\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r<\Theta_r<\mathfrak{Z}\Theta_r<\Gamma_r,\quad r\in\mathbb{N},$$

where

$$\mathfrak{Z}=\max\left\{\left[\Xi_{\mathfrak{z}_1}\prod_{i=1}^n\delta_i\int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1}\aleph_0(\tau,\tau)\nabla\tau\right]^{-1},\ 1\right\}.$$

Assume that f *satisfies*

(C_5) $f_j(\varpi)\leq\varphi(\mathfrak{N}_3\Gamma_r)$ *for all* $t\in[0,1]_{\mathbb{T}}$, $0\leq\varpi\leq\Gamma_r$, *where*

$$\mathfrak{N}_3<\min\left\{\left[\Xi\,\|\aleph_0\|_{L^\infty_\nabla}\prod_{i=1}^n\left\|\varphi^{-1}(\chi_i)\right\|_{L^1_\nabla}\right]^{-1},\mathfrak{Z}\right\};$$

(C_6) $f_j(\varpi) \geq \varphi(\mathfrak{Z}\Theta_r)$ *for all* $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, $\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r \leq \varpi \leq \Theta_r$.

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions* $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\varpi_j^{[r]}(t) \geq 0$ *on* $[0,1]_{\mathbb{T}}$, $j = 1, 2, \dots, \ell$, *and* $r \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.1. So, we omit the details here. □

4. Examples

In this section, we present an example to check validity of our main results.

Example 4.1. Consider the following boundary value problem on $\mathbb{T} = [0,1]$

$$\left.\begin{aligned}&\varphi(\varpi_j''(t)) + \chi(t) f_j(\varpi_{j+1}(t)) = 0, \quad j = 1, 2,\ t \in [0,1],\\ &\varpi_3(t) = \varpi_1(t),\end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned}\varpi_j(0) - \varpi_j'(0) &= \int_0^1 \frac{1}{2}\varpi_j(\tau)d\tau,\\ \varpi_j(1) + \varpi_j'(1) &= \int_0^1 \frac{1}{2}\varpi_j(\tau)d\tau,\end{aligned}\right\} \tag{4.2}$$

where

$$\varphi(\varpi) = \begin{cases} \dfrac{\varpi^3}{1+\varpi^2}, & \varpi \leq 0,\\ \varpi^2, & \varpi > 0,\end{cases}$$

$$\chi(t) = \chi_1(t) \cdot \chi_2(t),$$

in which

$$\chi_1(t) = \frac{1}{|t - \frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \chi_2(t) = \frac{1}{|t - \frac{1}{3}|^{\frac{1}{2}}},$$

$$f_1(\varpi) = f_2(\varpi)$$

$$= \begin{cases} 0.05 \times 10^{-8}, & \varpi \in (10^{-4}, +\infty),\\ \frac{5604\times 10^{-(8r+6)} - 0.05\times 10^{-8r}}{10^{-(4r+3)} - 10^{-4r}}(\varpi - 10^{-4r}) & \\ +0.05 \times 10^{-8r}, & \varpi \in \left[10^{-(4r+3)}, 10^{-4r}\right],\\ 5604 \times 10^{-(8r+6)}, & \varpi \in \left(0.98 \times 10^{-(4r+3)}, 10^{-(4r+3)}\right),\\ \frac{5604\times 10^{-(8r+6)} - 0.05\times 10^{-8r}}{0.98\times 10^{-(4r+3)} - 10^{-(4r+4)}}(\varpi - 10^{-(4r+4)}) & \\ +0.05 \times 10^{-8r}, & \varpi \in \left(10^{-(4r+4)}, 0.98 \times 10^{-(4r+3)}\right].\end{cases}$$

Let

$$t_r = \frac{31}{64} - \sum_{k=1}^{r} \frac{1}{4(k+1)^4}, \quad \mathfrak{z}_r = \frac{1}{2}(t_r + t_{r+1}), \quad \text{for } r = 1, 2, 3, \dots,$$

then

$$\mathfrak{z}_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32} \quad \text{and} \quad t_{r+1} < \mathfrak{z}_r < t_r, \quad \mathfrak{z}_r > \frac{1}{5}, \quad \text{for } r = 1, 2, 3, \dots$$

Therefore, $\frac{\mathfrak{z}_r}{1} = \mathfrak{z}_r > \frac{1}{5}$, $j = 1, 2, 3, \dots$ It is clear that

$$t_1 = \frac{15}{32} < \frac{1}{2}, \quad t_r - t_{r+1} = \frac{1}{4(r+2)^4}, \quad r = 1, 2, 3, \dots$$

Since $\sum_{x=1}^{\infty} \frac{1}{x^4} = \frac{\pi^4}{90}$ and $\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{r\to\infty} t_r = \frac{31}{64} - \sum_{k=1}^{\infty} \frac{1}{4(r+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} = 0.4637941914,$$

$\chi_1, \chi_2 \in L^p[0,1]$ for $0 < p < 2$, so $\delta_1 = \delta_2 = \frac{1}{\sqrt{3}}$,

$$a(t) = 2 - t, \quad b(t) = 1 + t, \quad d = 3, \quad \aleph_0(t, \tau) = \frac{1}{3} \begin{cases} (2-\tau)(1+t), & t \le \tau, \\ (2-t)(1+\tau), & \tau \le t, \end{cases}$$

$$c_i = \int_0^1 \left[\int_0^1 \aleph_0(\tau_1, \tau_2) \kappa_i(\tau_1) \nabla\tau_1 \right] \chi(\tau_2) \nabla\tau_2 = 2.774076198,$$

$$u_a = u_b = v_a = v_b = \frac{1}{4}, \quad \kappa_1^* = \kappa_2^* = \frac{1}{2}, \quad \kappa_1(\mathfrak{z}_1) = \kappa_2(\mathfrak{z}_1) = 0.06558641976,$$

$$\mathcal{L}(\mathfrak{z}_1) = \min\left\{ \frac{\alpha\mathfrak{z}_1 + \beta}{\alpha + \beta}, \frac{\gamma\mathfrak{z}_1 + \delta}{\gamma + \delta} \right\} = \frac{1 + \mathfrak{z}_1}{2} = 0.7336033950,$$

$$\eta(t) = \frac{(1 - v_b)a(t) + v_a b(t)}{d[(1-u_a)(1-v_b) - u_b v_a]} = \frac{7 - 2t}{6}, \quad \eta^* = \frac{7}{6}, \quad \eta(\mathfrak{z}_1) = 1.010931070,$$

$$\lambda(t) = \frac{(1 - u_a)b(t) + u_b a(t)}{d[(1-u_a)(1-v_b) - u_b v_a]} = \frac{5 - 2t}{6}, \quad \lambda^* = \frac{5}{6}, \quad \lambda(\mathfrak{z}_1) = 0.6775977366,$$

$$\Xi = 1 + \eta^* \kappa_1^* + \lambda^* \kappa_2^* = 2,$$

$$\Xi_{\mathfrak{z}_1} = \mathcal{L}(\mathfrak{z}_1)\Big[1 + \eta(\mathfrak{z}_1)\kappa_1(\mathfrak{z}_1) + \lambda(\mathfrak{z}_1)\kappa_2(\mathfrak{z}_1)\Big] = 0.8148459802.$$

Note that $\Xi_{\mathfrak{z}}$ is increasing, it follows that $1.969391539 = \Xi_{\mathfrak{z}_\infty} < \Xi_{\mathfrak{z}_r} < \Xi_{\mathfrak{z}_1} = 2$, $0.9846957695 \le \frac{\Xi_{\mathfrak{z}_r}}{\Xi} \le 2$ and

$$\int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau, \tau) \nabla\tau = \int_{\frac{15}{32} - \frac{1}{648}}^{1 - \frac{15}{32} + \frac{1}{648}} \frac{(2-\tau)(1+\tau)}{3} d\tau = 0.04918197800.$$

Thus, we get

$$\mathfrak{Z} = \max\left\{ \left[\Xi_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau \right]^{-1}, \ 1 \right\} = \max\Big\{74.85826138, 1\Big\}$$
$$= 74.85826138$$

and

$$\|\aleph_0\|_{L^q_\nabla} = \left[\int_0^1 |\aleph_0(\tau,\tau)|^q d\tau \right]^{\frac{1}{q}} < 1, \quad \text{for } 0 < q < 2.$$

Next, let $0 < \mathfrak{a} < 1$ be fixed. Then $\chi_1, \chi_2 \in L^{1+\mathfrak{a}}[0,1]$. It follows that

$$\|\varphi^{-1}(\chi_1)\|_{1+\mathfrak{a}} = \left[\frac{1}{3 - \mathfrak{a}} \left(3^{\frac{3-\mathfrak{a}}{4}} + 1 \right) 2^{\frac{1+\mathfrak{a}}{2}} \right]^{\frac{1}{1+\mathfrak{a}}}$$

and

$$\|\varphi^{-1}(\chi_2)\|_{1+\mathfrak{a}} = \left[\frac{4}{3-\mathfrak{a}}\left(2^{\frac{3-\mathfrak{a}}{4}}+1\right)(1/3)^{\frac{3-\mathfrak{a}}{4}}\right]^{\frac{1}{1+\mathfrak{a}}}.$$

So, for $0 < \mathfrak{a} < 1$, we have

$$0.2509961333 \leq \left[\Xi \left\|\aleph_0\right\|_{L^q_\nabla} \prod_{i=1}^{n} \left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla}\right]^{-1} \leq 0.2856331500.$$

Taking $\mathfrak{N}_1 = 0.2$. In addition, if we take

$$\Gamma_r = 10^{-4r}, \quad \Theta_r = 10^{-(4r+3)},$$

then

$$\begin{aligned}\Gamma_{r+1} = 10^{-(4r+4)} &< 0.9846957695 \times 10^{-(4r+3)} < \frac{\Xi_{\mathfrak{z}r}}{\Xi}\Theta_r < \Theta_r = 10^{-(4r+3)}\\ &< \Gamma_r = 10^{-4r},\\ \mathfrak{Z}\Theta_r &= 74.85826138 \times 10^{-(4r+3)} < 0.2 \times 10^{-4r} = \mathfrak{N}_1\Gamma_r, \quad r = 1,2,3,\ldots,\end{aligned}$$

and f_1, f_2 satisfies the following growth conditions:

$$\begin{aligned} f_1(\varpi) = f_2(\varpi) &\leq \varphi(\mathfrak{N}_1\Gamma_r) = \mathfrak{N}_1^2\Gamma_r^2 = 0.04 \times 10^{-8r}, \quad \varpi \in \left[0, 10^{-4r}\right]\\ f_1(\varpi) = f_2(\varpi) &\geq \varphi(\mathfrak{Z}\Theta_r) = \mathfrak{Z}^2\Theta_r^2\\ &= 5603.759297 \times 10^{-(8r+6)}, \quad \varpi \in \left[0.98 \times 10^{-(4r+3)}, 10^{-(4r+3)}\right].\end{aligned}$$

Then all the conditions of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the iterative boundary value problem (4.1)–(4.2) has denumerably many solutions $\left\{\left(\varpi_1^{[r]}, \varpi_2^{[r]}\right)\right\}_{r=1}^{\infty}$ such that $\varpi_j^{[r]}(t) \geq 0$ on $[0,1]$, $j = 1,2$ and $r \in \mathbb{N}$.

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[1]Department of Applied Mathematics,
College of Science and Technology,
Andhra University, Visakhapatnam,
India-530003
Email address: rajendra92@rediffmail.com

[2]Department of Applied Mathematics,
College of Science and Technology,
Andhra University, Visakhapatnam,
India-530003
Email address: khuddush89@gmail.com

[3]Government Degree College for Women
Marripalem, Koyyuru Mandal
Visakhapatnam, India-531116
Email address: vidyavijaya08@gmail.com